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A blow-up criterion of strong solution to a 3D viscous liquid–gas two-phase flow model with vacuum

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Abstract

In this paper, we get a unique local strong solution to a 3D viscous liquid–gas two-phase flow model in a smooth bounded domain. Besides, a blow-up criterion of the strong solution for $\frac{25}{3}\mu > \lambda$ is obtained. The method can be applied to study a blow-up criterion of the strong solution to Navier–Stokes equations for $\frac{25}{3}\mu > \lambda$, which improves the corresponding result about Navier–Stokes equations in Sun et al. (2011) [15] where $7\mu > \lambda$. Moreover, all the results permit the appearance of vacuum.

Résumé

Dans cet article, on démontre l'existence d'une solution classique tridimensionelle pour un modèle d'écoulement visqueux diphasique (liquide–gaz) dans un domaine borné et régulier. De plus, on établit un critère d'explosion de la solution lorsque $\frac{25}{3}\mu > \lambda$. La même méthode s'applique pour établir l'explosion de solution classique des équations de Navier–Stokes lorsque $\frac{25}{3}\mu > \lambda$, ce qui améliore le résultat correspondant obtenu dans Sun et al. (2011) [15] où l'explosion est établie pour $7\mu > \lambda$. Enfin, tous les résultats restent valables en présence de vide.

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Keywords: Liquid–gas two-phase flow model; Local strong solution; Blow-up criterion; Vacuum

1. Introduction

In this paper, we consider the following 3D viscous liquid–gas two-phase flow model

$$\begin{cases} m_t + \operatorname{div}(mu) = 0, \\ n_t + \operatorname{div}(nu) = 0, \\ (mu)_t + \operatorname{div}(mu \otimes u) + \nabla P(m,n) = \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u, \quad \text{in } \Omega \times (0,\infty), \end{cases} \tag{1.1}$$

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with the initial and boundary conditions

$$(m, n, u)|_{t=0} = (m_0, n_0, u_0), \quad \text{in } \overline{\Omega}, \tag{1.2}$$

$$u(x, t) = 0, \quad \text{on } \partial\Omega \times [0, \infty), \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain. Here $m = \alpha_l \rho_l$ and $n = \alpha_g \rho_g$ denote the liquid mass and gas mass, respectively; μ, λ are viscosity constants, satisfying

$$\mu > 0, \qquad 2\mu + 3\lambda \geqslant 0, \tag{1.4}$$

which implies $\mu + \lambda \geqslant \frac{1}{3}\mu > 0$.

The unknown variables α_l, $\alpha_g \in [0, 1]$ denote respectively the liquid and gas volume fractions, satisfying the fundamental relation: $\alpha_l + \alpha_g = 1$. Furthermore, the other unknown variables ρ_l and ρ_g denote respectively the liquid and gas densities, satisfying equations of state: $\rho_l = \rho_{l,0} + \frac{P - P_{l,0}}{a_l^2}$, $\rho_g = \frac{P}{a_g^2}$, where a_l, a_g are sonic speeds, respectively, in the liquid and gas, and $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants; u denotes velocity of the liquid and gas; P is the common pressure for both phases, which satisfies

$$P(m, n) = C^0\bigl(-b(m, n) + \sqrt{b(m, n)^2 + c(n)}\bigr), \tag{1.5}$$

with $C^0 = \frac{1}{2}a_l^2$, $k_0 = \rho_{l,0} - \frac{P_{l,0}}{a_l^2} > 0$, $a_0 = (\frac{a_g}{a_l})^2$, and

$$b(m, n) = k_0 - m - \left(\frac{a_g}{a_l}\right)^2 n = k_0 - m - a_0 n,$$
$$c(n) = 4k_0\left(\frac{a_g}{a_l}\right)^2 n = 4k_0 a_0 n.$$

For more information about the above models, please refer to [11,14,20] and references therein.

The investigation of model (1.1) has been a topic during the last decade. There are many results about the numerical properties of this model or related model. However, there are few results providing insight into existence, uniqueness, regularity, asymptotic behavior and decay rate estimates concerning the two-phase liquid–gas models of the form (1.1). Let us review some previous works about the viscous liquid–gas two-phase flow model. For the model (1.1) in 1D, when the liquid is incompressible and the gas is polytropic, i.e., $P(m, n) = C\rho_l^\gamma(\frac{n}{\rho_l - m})^\gamma$, Evje and Karlsen in [4] studied the existence and uniqueness of the global weak solution to the free boundary value problem with $\mu = \mu(m) = k_1 \frac{m^\beta}{(\rho_l - m)^{\beta+1}}$, $\beta \in (0, \frac{1}{3})$, when the fluids connected to vacuum state discontinuously. Yao and Zhu extended the results in [4] to the case $\beta \in (0, 1]$, and also obtained the asymptotic behavior and regularity of the solution, see [18]. Evje, Flåtten and Friis in [2] also studied the model with $\mu = \mu(m, n) = k_2 \frac{n^\beta}{(\rho_l - m)^{\beta+1}}$ ($\beta \in (0, \frac{1}{3})$) in a free boundary setting when the fluids connected to vacuum state continuously, and obtained the global existence of the weak solution. Also, for the case of connecting to vacuum state continuously, Yao and Zhu investigated the free boundary problem to the model with constant viscosity coefficient, and obtained the existence and uniqueness of the global weak solution by the line method, where a new technique was introduced to get the key upper and lower bounds of gas and liquid masses n and m, cf. [19]. Specifically, when both of the two fluids are compressible, one can consult Ref. [3]. For multidimensional case, the results are few. Recently, Yao, Zhang and Zhu obtained the existence of the global weak solution to the 2D model when the initial energy is small, see [20]. Furthermore, they proved a blow-up criterion in terms of the upper bound of the liquid mass for the strong solution to the 2D model in a smooth bounded domain, cf. [21]. Because of the complexity of the pressure $P(m, n)$, they in [21] can only deal with the case: there is no initial vacuum, i.e., $m_0 > 0$, $n_0 > 0$. Then, what will happen when the vacuum appears? In this paper, we prove the local existence of strong solution and give a blow-up criterion to the 3D viscous liquid–gas two-phase flow model in a smooth bounded domain with vacuum.

The main results are stated as follows:

Theorem 1.1 *(Local existence). Let Ω be a bounded smooth domain in $\mathbb{R}^3$ and $q \in (3, 6]$. Assume that the initial data m_0, n_0, u_0 satisfy $m_0, n_0 \in W^{1,q}(\Omega)$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $0 \leqslant \underline{s}_0 m_0 \leqslant n_0 \leqslant \bar{s}_0 m_0$ in $\overline{\Omega}$, where $\underline{s}_0$ and $\bar{s}_0$ are positive constants. The following compatible condition is also valid*:

$$-\mu \Delta u_0 - (\mu + \lambda)\nabla \operatorname{div} u_0 + \nabla P(m_0, n_0) = \sqrt{m_0} g, \quad \text{for some } g \in L^2(\Omega). \tag{1.6}$$

Then, there exist a $T_0 > 0$ and a unique strong solution (m, n, u) to the problem (1.1)–(1.5)*, such that*

$$\begin{gathered} 0 \leqslant \underline{s}_0 m \leqslant n \leqslant \bar{s}_0 m, \qquad (m, n) \in C\big([0, T_0]; W^{1,q}(\Omega)\big), \qquad (m_t, n_t) \in L^\infty\big(0, T_0; L^q(\Omega)\big), \\ P \in L^\infty\big(0, T_0; W^{1,q}(\Omega)\big), \qquad u \in L^\infty\big(0, T_0; H_0^1(\Omega) \cap H^2(\Omega)\big) \cap L^2\big(0, T_0; W^{2,q}(\Omega)\big), \\ \sqrt{m} u_t \in L^\infty\big(0, T_0; L^2(\Omega)\big), \qquad u_t \in L^2\big(0, T_0; H_0^1(\Omega)\big). \end{gathered} \tag{1.7}$$

Furthermore, under the assumption

$$\lambda < \frac{25}{3}\mu, \tag{1.8}$$

we can establish a blow-up criterion of the strong solution:

Theorem 1.2. *Under the assumptions of Theorem 1.1, if $T^* < \infty$ is the maximal existence time for the strong solution $(m, n, u)(x, t)$ to the problem* (1.1)–(1.5) *stated in Theorem* 1.1*, then*

$$\lim_{T \to T^*} \sup \|m\|_{L^\infty(0,T;L^\infty(\Omega))} = \infty, \tag{1.9}$$

provided that (1.8) *holds.*

Remark 1.3. (i) For $\Omega = \mathbb{R}^3$, we can also get a unique strong solution to (1.1)–(1.5) and the blow-up criterion (1.9) by using the ideas of [1,15] to modify the proofs of Theorem 1.1 and Theorem 1.2 slightly.

(ii) The proof of Theorem 1.1 and Theorem 1.2 implies that the following blow-up criterion would be obtained if the restriction (1.8) is removed:

$$\lim_{T \to T^*} \sup \big(\|m\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\sqrt{m} u\|_{L^s(0,T;L^{s'}(\Omega))}\big) = \infty, \tag{1.10}$$

where $\frac{2}{s} + \frac{3}{s'} \leqslant 1$ and $3 < s' \leqslant \infty$. (1.10) is similar to [7].

(iii) Under the assumption (1.8), we can use our methods in Lemma 5.2 together with the estimates in [15] to get the following blow-up criterion of strong solution to Navier–Stokes equations:

$$\lim_{T \to T^*} \sup \big\|\rho(t)\big\|_{L^\infty(0,T;L^\infty(\Omega))} = \infty.$$

This relaxes the restriction $7\mu > \lambda$ in [15]. And our result can be viewed to be a generalization of [15].

We should mention that the methods introduced by Hoff in [6], Sun, Wang and Zhang in [15], Cho, Choe and Kim in [1] for the Navier–Stokes equations will play a crucial role in our proof here. There are many results about blow-up criterion of the strong solution for the Navier–Stokes equations in addition to [7]. For the 2D compressible Navier–Stokes equations, Sun and Zhang in [16] obtained a blow-up criterion in terms of the upper bound of the density for the strong solution. For the 3D compressible Navier–Stokes equations, Sun, Wang and Zhang in [15] obtained a blow-up criterion in terms of the upper bound of the density for the strong solution, under the restriction $\lambda < 7\mu$. In both papers, the initial vacuum ($\rho_0 \geqslant 0$) was allowed and the domain included both the bounded smooth domain and $\mathbb{R}^N$, $N = 2, 3$. It also worths mentioning recent works [8,9]. Under the assumptions

$$N = 2, \qquad \mu > 0, \qquad \mu + \lambda \geqslant 0, \qquad \Omega = T^2,$$

or

$$N = 3, \qquad \lambda < 7\mu, \qquad \mu > 0, \quad \text{and} \quad 2\mu + 3\lambda \geqslant 0, \qquad \Omega \text{ is a smooth domain including } \mathbb{R}^3,$$

Huang and Xin proved the following blow-up criterion: if $T^* < \infty$ is the maximal time of the existence of the strong solution, then

$$\lim_{T\to T^*}\int_0^T \big\|\nabla u(t)\big\|_{L^\infty(\Omega)}\,dt = \infty. \tag{1.11}$$

Huang, Li and Xin in their recent paper [10] removed the restriction $\lambda < 7\mu$ for $N = 3$, and got the blow-up criterion of strong solution:

$$\lim_{T\to T^*}\int_0^T \big\|\mathcal{D}(u)(t)\big\|_{L^\infty(\Omega)}\,dt = \infty,$$

where $\mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla u^t)$. For the non-isentropic compressible Navier–Stokes equations, under the conditions: $N = 2$, $\mu > 0$, $\mu + \lambda \geqslant 0$, $\Omega = T^2$ or $[0,1]^2$; $N = 3$, $\lambda < 7\mu$, $\mu > 0$, and $2\mu + 3\lambda \geqslant 0$, Ω is a smooth bounded domain, please refer to [12,5].

In Theorem 1.2, we give a blow-up criterion in terms of the upper bound of the liquid mass under the relaxed restriction (1.8), which improves the corresponding result about Navier–Stokes equations in [15] where $7\mu > \lambda$. Here, if the liquid mass is upper bounded, we can obtain a high integrability of the velocity, $\sup_{0\leqslant t\leqslant T}\int_\Omega m|u|^r\,dx \leqslant C$, for some $r \in (3,4]$, see Lemma 5.2. Moreover, in order to overcome the singularity brought by the pressure $P(m,n)$ when there is vacuum, we need the assumption: $0 \leqslant \underline{s}_0 m_0 \leqslant n_0 \leqslant \bar{s}_0 m_0$, where $\underline{s}_0$ and $\bar{s}_0$ are positive constants.

2. Preliminaries

In this section, we give some useful lemmas which will be used in the next three sections, where $N = 2, 3$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function $u \in W_0^{1,p}(\Omega)$ or $u \in W^{1,p}(\Omega)$, $\int_\Omega u\,dx = 0$:*

$$\|u\|_{L^{p'}(\Omega)} \leqslant C_1\|\nabla u\|^{\alpha}_{L^p(\Omega)}\|u\|^{1-\alpha}_{L^{r'}(\Omega)}, \tag{2.1}$$

where $\alpha = (1/r' - 1/p')(1/r' - 1/p + 1/N)^{-1}$; moreover, if $p < N$, then $p' \in [r', pN/(N-p)]$ for $r' \leqslant pN/(N-p)$, and $p' \in [pN/(N-p), r']$ for $r' > pN/(N-p)$. If $p \geqslant N$, then $p' \in [r', \infty)$ is arbitrary; moreover, if $p > N$, then inequality (2.1) *is also valid for $p' = \infty$. The positive constant C_1 in inequality* (2.1) *depends on N, p, r', α and the domain Ω but independent of the function u.*

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function $u \in W^{1,p}(\Omega)$:*

$$\|u\|_{L^{p'}(\Omega)} \leqslant C_2\big(\|u\|_{L^1(\Omega)} + \|\nabla u\|^{\alpha}_{L^p(\Omega)}\|u\|^{1-\alpha}_{L^{r'}(\Omega)}\big), \tag{2.2}$$

where N, p, r', p' and α are the same as those in Lemma 2.1. *The positive constant C_2 in inequality* (2.2) *depends on N, p, r', α and the domain Ω but independent of the function u.*

The above two lemmas can be found in [13,17] and the references therein.

Next, we give some L^p $(p \in (1,\infty))$ regularity estimates for the solution of the following boundary problem:

$$\begin{cases} LU := \mu\Delta U + (\mu+\lambda)\nabla\operatorname{div}U = F, & \text{in } \Omega,\\ U(x) = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, L is the Lamé operator, $U = (U_1, U_2, \dots, U_N)$, $F = (F_1, F_2, \dots, F_N)$. From (1.4), we know that (2.3) is a strong elliptic system. If $F \in W^{-1,2}(\Omega)$, then there exists an unique weak solution $U \in H_0^1(\Omega)$. In the subsequent context, we will use $L^{-1}F$ to denote the unique solution U of the system (2.3) with F belonging to some suitable space such as $W^{-1,p}(\Omega)$. Sun, Wang and Zhang in [15,16] give the following estimates:

Lemma 2.3. *Let $p \in (1, \infty)$, and U be a solution of* (2.3). *Then there exists a constant C depending only on μ, λ, p, N and Ω such that*

(1) *if $F \in L^p(\Omega)$, then*

$$\|U\|_{W^{2,p}(\Omega)} \leqslant C\|F\|_{L^p(\Omega)}; \tag{2.4}$$

(2) *if $F \in W^{-1,p}(\Omega)$ (i.e., $F = \operatorname{div} f$ with $f = (f_{ij})_{N\times N}$, $f_{ij} \in L^p(\Omega)$), then*

$$\|U\|_{W^{1,p}(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}; \tag{2.5}$$

(3) *if $F = \operatorname{div} f$ with $f_{ij} = \partial_k h^k_{ij}$ and $h^k_{ij} \in W^{1,p}_0(\Omega)$ for $i, j, k = 1, 2, \dots, N$, then*

$$\|U\|_{L^p(\Omega)} \leqslant C\|h\|_{L^p(\Omega)}. \tag{2.6}$$

Lemma 2.4. *If $F = \operatorname{div} f$ with $f = (f_{ij})_{N\times N}$, $f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega)$, then $\nabla U \in BMO(\Omega)$ and there exists a constant C depending only on μ, λ and Ω such that*

$$\|\nabla U\|_{BMO(\Omega)} \leqslant C\big(\|f\|_{L^\infty(\Omega)} + \|f\|_{L^2(\Omega)}\big). \tag{2.7}$$

Here $BMO(\Omega)$ denotes the John–Nirenberg's space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO(\Omega)} = \|f\|_{L^2(\Omega)} + [f]_{BMO(\Omega)},$$

with the semi-norm

$$[f]_{BMO(\Omega)} = \sup_{x\in\Omega, r\in(0,d)} \fint_{\Omega_r(x)} \big|f(y) - f_{\Omega_r(x)}\big|\, dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center x and radius r and d is the diameter of Ω. For a measurable subset E of $\mathbb{R}^N$, $|E|$ denotes its Lebesgue measure, and

$$f_{\Omega_r(x)} = \fint_{\Omega_r(x)} f(y)\, dy = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y)\, dy.$$

Lemma 2.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $f \in W^{1,p}(\Omega)$ with $p \in (N, \infty)$. Then there exists a constant C depending on p, N and the Lipschitz property of the domain Ω such that*

$$\|f\|_{L^\infty(\Omega)} \leqslant C\big(1 + \|f\|_{BMO(\Omega)} \ln\big(e + \|\nabla f\|_{L^p(\Omega)}\big)\big). \tag{2.8}$$

3. Global existence for the linearized system

Consider

$$\begin{cases} m_t + \operatorname{div}(mv) = 0, \\ n_t + \operatorname{div}(nv) = 0, \\ mu_t + mv \cdot \nabla u + \nabla P(m, n) = Lu, \quad \text{in } \Omega \times (0, \infty), \end{cases} \tag{3.1}$$

with the initial and boundary conditions

$$(m, n, u)|_{t=0} = (m_0, n_0, u_0), \quad \text{in } \overline{\Omega}, \tag{3.2}$$

$$u(x, t) = 0, \quad \text{on } \partial\Omega \times [0, \infty), \tag{3.3}$$

where $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain.

Throughout the rest of the paper, we denote $W^{k,p} = W^{k,p}(\Omega)$ for $k \geqslant 0$ and $1 < p \leqslant \infty$ with the norm $\|\cdot\|_{W^{k,p}}$. Particularly, $H^k = W^{k,2}$, and $L^p = W^{0,p}$. $Q_T = \overline{\Omega} \times [0, T]$.

Theorem 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and $q \in (3,6]$. Assume m_0, $n_0 \in W^{1,q}$, $u_0 \in H_0^1 \cap H^2$, $m_0 \geqslant \delta > 0$, $n_0 \geqslant \delta > 0$, $v \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; W^{2,q})$ and $v_t \in L^2(0,T; H_0^1)$. Then there exists a unique strong solution (m,n,u) to* (3.1)–(3.3) *such that*

$$\begin{gathered}
(m,n) \in C\big([0,T]; W^{1,q}\big), \qquad (m_t,n_t) \in C\big([0,T]; L^q\big),\\
P \in C\big([0,T]; W^{1,q}\big), \qquad u \in C\big([0,T]; H_0^1 \cap H^2\big) \cap L^2\big(0,T; W^{2,q}\big),\\
u_t \in C\big([0,T]; L^2\big) \cap L^2\big(0,T; H_0^1\big), \qquad m > 0, \quad n > 0 \quad \text{in } Q_T.
\end{gathered}$$

Proof. By [1], $(3.1)_1$ and $(3.1)_2$, we get

$$\begin{cases}
m,n \in C\big([0,T]; W^{1,q}\big); \qquad m_t,n_t \in C\big([0,T]; L^q\big),\\
\sup\limits_{0\leqslant t\leqslant T} \big\|m(t)\big\|_{W^{1,q}} \leqslant \|m_0\|_{W^{1,q}} \exp\left\{C\displaystyle\int_0^T \big\|\nabla v(s)\big\|_{W^{1,q}}\,ds\right\},\\
\sup\limits_{0\leqslant t\leqslant T} \big\|n(t)\big\|_{W^{1,q}} \leqslant \|n_0\|_{W^{1,q}} \exp\left\{C\displaystyle\int_0^T \big\|\nabla v(s)\big\|_{W^{1,q}}\,ds\right\},\\
0 < \delta \exp\left\{-\displaystyle\int_0^T \big\|\nabla v(s)\big\|_{L^\infty}\,ds\right\} \leqslant m \leqslant \|m_0\|_{L^\infty} \exp\left\{\displaystyle\int_0^T \big\|\nabla v(s)\big\|_{L^\infty}\,ds\right\},\\
0 < \delta \exp\left\{-\displaystyle\int_0^T \big\|\nabla v(s)\big\|_{L^\infty}\,ds\right\} \leqslant n \leqslant \|n_0\|_{L^\infty} \exp\left\{\displaystyle\int_0^T \big\|\nabla v(s)\big\|_{L^\infty}\,ds\right\}.
\end{cases} \tag{3.4}$$

This immediately gives

$$P(m,n) \in C\big([0,T]; W^{1,q}\big), \qquad P(m,n)_t \in C\big([0,T]; L^q\big). \tag{3.5}$$

It follows from $(3.1)_3$, (3.4), (3.5) and [1] that

$$u \in C\big([0,T]; H_0^1 \cap H^2\big) \cap L^2\big(0,T; W^{2,q}\big), \qquad u_t \in C\big([0,T]; L^2\big) \cap L^2\big(0,T; H_0^1\big). \qquad \square$$

4. Proof of Theorem 1.1

In this section, we get a unique local strong solution to (1.1)–(1.5) with $m_0 \geqslant \delta > 0$, $n_0 \geqslant \delta > 0$, and obtain some estimates uniformly for δ (see Theorem 4.1). Theorem 1.1 will be obtained after constructing a sequence of approximate solutions $(m^\delta, n^\delta, u^\delta)$ by giving the initial data (m_0, n_0) in Theorem 1.1 a lower bound δ, using the estimates in Theorem 4.1, and taking $\delta \to 0$ (taking subsequence if necessary).

Theorem 4.1. *Under the conditions of Theorem* 1.1, *we assume $m_0 \geqslant \delta > 0$, $n_0 \geqslant \delta > 0$. Then there exists a time $T_0 > 0$ independent of δ and a unique strong solution (m,n,u) to* (1.1)–(1.5) *such that*

$$\begin{gathered}
(m,n) \in C\big([0,T_0]; W^{1,q}\big), \qquad (m_t,n_t) \in C\big([0,T_0]; L^q\big),\\
P \in C\big([0,T_0]; W^{1,q}\big), \qquad u \in C\big([0,T_0]; H_0^1 \cap H^2\big) \cap L^2\big(0,T_0; W^{2,q}\big),\\
u_t \in C\big([0,T_0]; L^2\big) \cap L^2\big(0,T_0; H_0^1\big), \qquad m > 0, \quad n > 0 \quad \text{in } Q_{T_0}.
\end{gathered}$$

Moreover, we have the following estimates:

$$\begin{cases} \sup\limits_{0\leqslant t\leqslant T_0} \int\limits_{\Omega} m|u_t|^2 + \int\limits_0^{T_0}\int\limits_{\Omega} |\nabla u_t|^2 \leqslant C, \\ \|u\|_{L^2(0,T_0;W^{2,q})} + \|u\|_{L^\infty(0,T_0;H_0^1\cap H^2)} + \|m\|_{L^\infty(0,T_0;W^{1,q})} + \|n\|_{L^\infty(0,T_0;W^{1,q})} \leqslant C, \\ \frac{\underline{s}_0\delta}{C} \leqslant \underline{s}_0 m \leqslant n \leqslant \bar{s}_0 m, \quad \text{in } Q_{T_0}, \\ \|P\|_{L^\infty(Q_{T_0})} + \|P_m\|_{L^\infty(Q_{T_0})} + \|P_n\|_{L^\infty(Q_{T_0})} \leqslant C, \end{cases}$$

where C is a positive constant, independent of δ.

To prove this theorem, we first construct a sequence of approximate solutions inductively as follows (similar to [1]):
(i) Define $u^0 = 0$, and assume $u^{k-1} \in C([0,T]; H_0^1 \cap H^2) \cap L^2(0,T; W^{2,q}) \cap H^1(0,T; H_0^1)$ was defined for $k \geqslant 1$.
(ii) By Theorem 3.1, we can get (m^k, n^k, u^k) with the regularities in Theorem 3.1 satisfying

$$\begin{cases} m_t^k + \operatorname{div}(m^k u^{k-1}) = 0, \\ n_t^k + \operatorname{div}(n^k u^{k-1}) = 0, \\ m^k u_t^k + m^k u^{k-1} \cdot \nabla u^k + \nabla P^k = L u^k, \quad \text{in } \Omega \times (0,T], \end{cases} \tag{4.1}$$

where $P^k = P(m^k, n^k)$. The initial and boundary conditions are stated as follows

$$\left(m^k, n^k, u^k\right)\big|_{t=0} = (m_0, n_0, u_0), \quad \text{in } \overline{\Omega}, \tag{4.2}$$

$$u^k(x,t) = 0, \quad \text{on } \partial\Omega \times [0,T]. \tag{4.3}$$

Throughout this paper, we denote

$$\Phi_K(t) = \max_{1\leqslant k\leqslant K}\left(1 + \left\|m^k(t)\right\|_{L^\infty}\right), \qquad \Psi_{K,r}(t) = \max_{1\leqslant k\leqslant K}\left(1 + \int\limits_{\Omega} m^k \left|u^{k-1}\right|^r\right),$$

for $r \in (3,4]$ and $K \in Z_+$. The next step is to make some estimates for (m^k, n^k, u^k) $(k \geqslant 1)$ independent of k and δ.

Lemma 4.2. *Under the conditions of Theorem* 4.1*, we have for all* $k \geqslant 1$

$$0 < \underline{s}_0 m^k \leqslant n^k \leqslant \bar{s}_0 m^k, \quad \textit{in } Q_T.$$

Proof. It follows from $(4.1)_1$ and $(4.1)_2$ that

$$\left(\frac{n^k}{m^k}\right)_t + u^{k-1} \cdot \nabla\left(\frac{n^k}{m^k}\right) = 0.$$

This implies

$$\frac{d}{ds}\left(\frac{n^k}{m^k}\right)\left(X(s;x,t),s\right) = 0, \tag{4.4}$$

where $X(s;x,t)$ is given by:

$$\begin{cases} \frac{d}{ds} X(s;x,t) = u^{k-1}\left(X(s;x,t),s\right), \quad 0 \leqslant s < t, \\ X(t;x,t) = x. \end{cases}$$

Integrating (4.4) over $(0,t)$, and using the assumption $\underline{s}_0 m_0 \leqslant n_0 \leqslant \bar{s}_0 m_0$, we complete the proof of Lemma 4.2. □

Lemma 4.3. *Under the conditions of Theorem* 4.1*, we have for all* $1 \leqslant k \leqslant K$

$$0 < P^k \leqslant C\Phi_K(t), \quad \textit{in } Q_T, \tag{4.5}$$

$$0 < P^k_{m^k} \leqslant C, \quad \textit{in } Q_T, \tag{4.6}$$

$$0 < P^k_{n^k} \leqslant C, \quad \text{in } Q_T, \tag{4.7}$$

where C is a positive constant, independent of K, δ and T.

Proof. (4.5) can be obtained by Lemma 4.2 and (1.5). A direct calculation gives

$$P^k_{m^k} = C^0\left\{1 - \frac{b(m^k, n^k)}{\sqrt{b^2(m^k, n^k) + c(n^k)}}\right\} > 0, \tag{4.8}$$

$$P^k_{n^k} = C^0\left\{a_0 + \frac{a_0}{\sqrt{b^2(m^k, n^k) + c(n^k)}}\left(m^k + a_0 n^k + k_0\right)\right\} > 0. \tag{4.9}$$

Obviously, we get (4.6) by (4.8). To get (4.7), it suffices to prove

$$\left(\frac{m^k + a_0 n^k + k_0}{\sqrt{b^2(m^k, n^k) + c(n^k)}}\right)^2 \leqslant C.$$

In fact,

$$\begin{aligned}
\left(\frac{m^k + a_0 n^k + k_0}{\sqrt{b^2(m^k, n^k) + c(n^k)}}\right)^2 &= \frac{(m^k)^2 + a_0^2(n^k)^2 + k_0^2 + 2k_0 m^k + 2a_0 m^k n^k + 2a_0 k_0 n^k}{(m^k)^2 + a_0^2(n^k)^2 + k_0^2 - 2k_0 m^k + 2a_0 m^k n^k + 2a_0 k_0 n^k} \\
&= 1 + \frac{4k_0 m^k}{(k_0 - m^k)^2 + a_0^2(n^k)^2 + 2a_0 m^k n^k + 2a_0 k_0 n^k} \\
&\leqslant 1 + \frac{4k_0 m^k}{2a_0 k_0 n^k} \\
&\leqslant C,
\end{aligned}$$

where we have used Lemma 4.2. This completes the proof of Lemma 4.3. □

As in [15], we denote $w^k = u^k - h^k$, where h^k is the unique solution to

$$\begin{cases} Lh^k = \nabla P^k, & \text{in } \Omega \times (0, T], \\ h^k|_{\partial\Omega} = 0. \end{cases} \tag{4.10}$$

From Lemma 2.3, we get for any $p \in (1, \infty)$

$$\begin{cases} \|h^k\|_{W^{1,p}} \leqslant C\|P^k\|_{L^p}, \\ \|h^k\|_{W^{2,p}} \leqslant C\|\nabla P^k\|_{L^p}; \end{cases} \tag{4.11}$$

$(4.1)_3$, (4.3) and (4.10) imply

$$\begin{cases} m^k w^k_t - Lw^k = m^k F^k, & \text{in } \Omega \times (0, T], \\ w^k(x, 0) = u_0 - L^{-1}\nabla P(m_0, n_0), & \text{in } \overline{\Omega}, \\ w^k|_{\partial\Omega} = 0, \end{cases} \tag{4.12}$$

where

$$\begin{aligned}
F^k &= -u^{k-1} \cdot \nabla u^k - L^{-1}\nabla P^k_t \\
&= -u^{k-1} \cdot \nabla u^k + L^{-1}\nabla \operatorname{div}\left(P^k u^{k-1}\right) + L^{-1}\nabla\left[\left(m^k P^k_{m^k} + n^k P^k_{n^k} - P^k\right)\operatorname{div} u^{k-1}\right].
\end{aligned}$$

Lemma 4.4. *Under the conditions of Theorem 4.1, we have for all $1 \leqslant k \leqslant K$, $3 < r \leqslant 4$*

$$\int_0^T \int_\Omega m^k |w^k_t|^2 + \int_\Omega |\nabla w^k|^2 \leqslant C \exp\left\{C \int_0^T \left[\Phi_K(s)\right]^{\frac{r+1}{r-3}} \left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}} ds\right\},$$

and

$$\int_0^T\int_\Omega \left|\nabla^2 w^k\right|^2 \leqslant C \sup_{0\leqslant t\leqslant T} \Phi_K(t)\exp\left\{C\int_0^T \left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\},$$

where C is a positive constant, independent of K, δ and T.

Proof. Multiplying (4.12) by w_t^k, integrating over Ω, and using integration by parts and Cauchy inequality, we have

$$\begin{aligned}
&\int_\Omega m^k\left|w_t^k\right|^2+\frac{1}{2}\frac{d}{dt}\int_\Omega\left[\mu\left|\nabla w^k\right|^2+(\mu+\lambda)\left|\operatorname{div} w^k\right|^2\right]\\
&\quad\leqslant\frac{1}{2}\int_\Omega m^k\left|w_t^k\right|^2+\frac{1}{2}\int_\Omega m^k\left|F^k\right|^2,
\end{aligned}$$

which implies

$$\int_\Omega m^k\left|w_t^k\right|^2+\frac{d}{dt}\int_\Omega\left[\mu\left|\nabla w^k\right|^2+(\mu+\lambda)\left|\operatorname{div} w^k\right|^2\right]\leqslant\int_\Omega m^k\left|F^k\right|^2. \tag{4.13}$$

Now we estimate the term of the right side in (4.13) as follows:

$$\begin{aligned}
\int_\Omega m^k\left|F^k\right|^2 &\leqslant C\int_\Omega m^k\left|u^{k-1}\right|^2\left|\nabla u^k\right|^2+C\Phi_K(t)\int_\Omega\left|L^{-1}\nabla\operatorname{div}\left(P^k u^{k-1}\right)\right|^2+C\Phi_K(t)\int_\Omega\left(\left|m^k\right|^2+\left|P^k\right|^2\right)\left|\operatorname{div} u^{k-1}\right|^2\\
&\leqslant C\left[\int_\Omega\left(m^k\left|u^{k-1}\right|^2\right)^{\frac{r}{2}}\right]^{\frac{2}{r}}\left[\int_\Omega\left|\nabla u^k\right|^{\frac{2r}{r-2}}\right]^{\frac{r-2}{r}}+C\Phi_K(t)\int_\Omega\left|P^k u^{k-1}\right|^2+C\left[\Phi_K(t)\right]^3\int_\Omega\left|\operatorname{div} u^{k-1}\right|^2\\
&\leqslant C\left[\int_\Omega\left(m^k\right)^{\frac{r}{2}}\left|u^{k-1}\right|^r\right]^{\frac{2}{r}}\left\|\nabla u^k\right\|^2_{L^{\frac{2r}{r-2}}}+C\left[\Phi_K(t)\right]^3\int_\Omega\left|\nabla u^{k-1}\right|^2+C\left[\Phi_K(t)\right]^3\int_\Omega\left|\operatorname{div} u^{k-1}\right|^2\\
&\leqslant C\left[\Phi_K(t)\right]^{\frac{r-2}{r}}\left[\Psi_{K,r}(t)\right]^{\frac{2}{r}}\left(\left\|\nabla w^k\right\|^2_{L^{\frac{2r}{r-2}}}+\left\|\nabla h^k\right\|^2_{L^{\frac{2r}{r-2}}}\right)\\
&\quad+C\left[\Phi_K(t)\right]^3\int_\Omega\left[\mu\left|\nabla u^{k-1}\right|^2+(\mu+\lambda)\left|\operatorname{div} u^{k-1}\right|^2\right]\\
&\leqslant C\left[\Phi_K(t)\right]^{\frac{r-2}{r}}\left[\Psi_{K,r}(t)\right]^{\frac{2}{r}}\left(\varepsilon\left\|\nabla^2 w^k\right\|^2_{L^2}+\left(\varepsilon^{\frac{-3}{r-3}}+1\right)\left\|\nabla w^k\right\|^2_{L^2}+\left[\Phi_K(t)\right]^2\right)\\
&\quad+C\left[\Phi_K(t)\right]^3\int_\Omega\left[\mu\left|\nabla u^{k-1}\right|^2+(\mu+\lambda)\left|\operatorname{div} u^{k-1}\right|^2\right],
\end{aligned}\tag{4.14}$$

where we have used Lemmas 2.2, 2.3, 4.2, 4.3 and Young inequality: $ab\leqslant\varepsilon a^p+(\varepsilon p)^{\frac{-q}{p}}q^{-1}b^q$ for any $\varepsilon>0$, $p,q>0$ and $\frac{1}{p}+\frac{1}{q}=1$.

By Lemma 2.3 and (4.12), we have

$$\left\|\nabla^2 w^k\right\|^2_{L^2}\leqslant C\Phi_K(t)\int_\Omega m^k\left|w_t^k\right|^2+C\Phi_K(t)\int_\Omega m^k\left|F^k\right|^2. \tag{4.15}$$

Substituting (4.15) into (4.14), we have

$$\begin{aligned}
\int_\Omega m^k\left|F^k\right|^2 &\leqslant C\varepsilon\left[\Phi_K(t)\right]^{\frac{2r-2}{r}}\left[\Psi_{K,r}(t)\right]^{\frac{2}{r}}\left(\int_\Omega m^k\left|w_t^k\right|^2+\int_\Omega m^k\left|F^k\right|^2\right)\\
&\quad+C\left[\Phi_K(t)\right]^{\frac{r-2}{r}}\left[\Psi_{K,r}(t)\right]^{\frac{2}{r}}\left(\varepsilon^{\frac{-3}{r-3}}+1\right)\int_\Omega\mu\left|\nabla w^k\right|^2+C\left[\Phi_K(t)\right]^{\frac{3r-2}{r}}\left[\Psi_{K,r}(t)\right]^{\frac{2}{r}}
\end{aligned}$$

$$
\begin{aligned}
&+ C\big[\Phi_K(t)\big]^3 \int_\Omega \big[\mu\big|\nabla w^{k-1}\big|^2 + (\mu+\lambda)\big|\operatorname{div} w^{k-1}\big|^2\big] \\
&+ C\big[\Phi_K(t)\big]^3 \int_\Omega \big[\mu\big|\nabla h^{k-1}\big|^2 + (\mu+\lambda)\big|\operatorname{div} h^{k-1}\big|^2\big] \\
\leqslant{}& C\varepsilon\big[\Phi_K(t)\big]^{\frac{2r-2}{r}}\big[\Psi_{K,r}(t)\big]^{\frac{2}{r}}\bigg(\int_\Omega m^k\big|w_t^k\big|^2 + \int_\Omega m^k\big|F^k\big|^2\bigg) \\
&+ C\big[\Phi_K(t)\big]^{\frac{r-2}{r}}\big[\Psi_{K,r}(t)\big]^{\frac{2}{r}}\big(\varepsilon^{\frac{-3}{r-3}}+1\big)\int_\Omega \mu\big|\nabla w^k\big|^2 + C\big[\Phi_K(t)\big]^{\frac{3r-2}{r}}\big[\Psi_{K,r}(t)\big]^{\frac{2}{r}} \\
&+ C\big[\Phi_K(t)\big]^3 \int_\Omega \big[\mu\big|\nabla w^{k-1}\big|^2 + (\mu+\lambda)\big|\operatorname{div} w^{k-1}\big|^2\big] + C\big[\Phi_K(t)\big]^5,
\end{aligned}
$$

where we have used (4.11).

Take $\varepsilon = \frac{1}{4C}[\Phi_K(t)]^{\frac{2-2r}{r}}[\Psi_{K,r}(t)]^{\frac{-2}{r}}$, we have

$$
\begin{aligned}
\int_\Omega m^k\big|F^k\big|^2 \leqslant{}& \frac{1}{3}\int_\Omega m^k\big|w_t^k\big|^2 + C\big[\Phi_K(t)\big]^{\frac{r+1}{r-3}}\big[\Psi_{K,r}(t)\big]^{\frac{2}{r-3}}\int_\Omega \mu\big|\nabla w^k\big|^2 + C\big[\Phi_K(t)\big]^5\big[\Psi_{K,r}(t)\big]^{\frac{2}{r}} \\
&+ C\big[\Phi_K(t)\big]^3 \int_\Omega \big[\mu\big|\nabla w^{k-1}\big|^2 + (\mu+\lambda)\big|\operatorname{div} w^{k-1}\big|^2\big]. \qquad (4.16)
\end{aligned}
$$

Combining (4.13) and (4.16), we get

$$
\begin{aligned}
&\frac{2}{3}\int_\Omega m^k\big|w_t^k\big|^2 + \frac{d}{dt}\int_\Omega \big[\mu\big|\nabla w^k\big|^2 + (\mu+\lambda)\big|\operatorname{div} w^k\big|^2\big] \\
&\quad\leqslant C\big[\Phi_K(t)\big]^{\frac{r+1}{r-3}}\big[\Psi_{K,r}(t)\big]^{\frac{2}{r-3}}\int_\Omega \mu\big|\nabla w^k\big|^2 + C\big[\Phi_K(t)\big]^5\big[\Psi_{K,r}(t)\big]^{\frac{2}{r}} \\
&\qquad+ C\big[\Phi_K(t)\big]^3 \int_\Omega \big[\mu\big|\nabla w^{k-1}\big|^2 + (\mu+\lambda)\big|\operatorname{div} w^{k-1}\big|^2\big].
\end{aligned}
$$

Integrating over $(0,t)$, we have

$$
\begin{aligned}
&\frac{2}{3}\int_0^t\int_\Omega m^k\big|w_t^k\big|^2 + \int_\Omega \big[\mu\big|\nabla w^k\big|^2 + (\mu+\lambda)\big|\operatorname{div} w^k\big|^2\big] \\
&\quad\leqslant C\int_0^t\big[\Phi_K(s)\big]^{\frac{r+1}{r-3}}\big[\Psi_{K,r}(s)\big]^{\frac{2}{r-3}}\int_\Omega \mu\big|\nabla w^k\big|^2 + C\int_0^t\big[\Phi_K(s)\big]^5\big[\Psi_{K,r}(s)\big]^{\frac{2}{r}} \\
&\qquad+ C\int_0^t\big[\Phi_K(s)\big]^3 \int_\Omega \big[\mu\big|\nabla w^{k-1}\big|^2 + (\mu+\lambda)\big|\operatorname{div} w^{k-1}\big|^2\big] + C. \qquad (4.17)
\end{aligned}
$$

Denote $A_K(t) = \sup_{1\leqslant k\leqslant K}\int_\Omega[\mu|\nabla w^k|^2 + (\mu+\lambda)|\operatorname{div} w^k|^2]$, we obtain from (4.17) noticing that $\frac{r+1}{r-3} > 3$ for $3 < r \leqslant 4$ and $\Phi_K \geqslant 1$, $\Psi_{K,r} \geqslant 1$

$$
A_K(t) \leqslant C\int_0^t\big[\Phi_K(s)\big]^{\frac{r+1}{r-3}}\big[\Psi_{K,r}(s)\big]^{\frac{2}{r-3}}A_K(s) + C\int_0^t\big[\Phi_K(s)\big]^5\big[\Psi_{K,r}(s)\big]^{\frac{2}{r}} + C.
$$

By Gronwall inequality, we get

$$
\begin{aligned}
A_K(t) &\leqslant C\exp\left\{C\int_0^t\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\}\\
&\quad+C\int_0^t\left[\Phi_K(\tau)\right]^5\left[\Psi_{K,r}(\tau)\right]^{\frac{2}{r}}\exp\left\{C\int_\tau^t\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\}d\tau\\
&\leqslant C\exp\left\{C\int_0^t\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\right\}ds, \qquad (4.18)
\end{aligned}
$$

where we have used the inequality: $y \leqslant \exp\{y\}$ for $y \geqslant 0$. By (4.15), (4.16), (4.17) and (4.18), we complete the proof of Lemma 4.4. □

From (4.11) and Lemma 4.4, we immediately give the following corollary:

Corollary 4.5. *Under the conditions of Theorem* 4.1*, we have for all* $1 \leqslant k \leqslant K$ *and* $3 < r \leqslant 4$

$$
\|\nabla u^k\|_{L^\infty(0,T;L^2)} \leqslant C\sup_{0\leqslant t\leqslant T}\Phi_K(t)\exp\left\{C\int_0^T\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\},
$$

and

$$
\|\nabla u^k\|_{L^2(0,T;L^6)} \leqslant C\Big[\sup_{0\leqslant t\leqslant T}\Phi_K(t)+\sqrt{T}\Big]\exp\left\{C\int_0^T\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\},
$$

where C *is a positive constant, independent of* K, δ *and* T.

Now we give higher order estimates for u^k.

Lemma 4.6. *Under the conditions of Theorem* 4.1*, we have for all* $1 \leqslant k \leqslant K$ *and* $3 < r \leqslant 4$

$$
\begin{aligned}
&\int_\Omega m^k\left|\dot{u}^k\right|^2+\int_0^t\int_\Omega\left(\mu\left|\nabla\dot{u}^k\right|^2+(\mu+\lambda)\left|\operatorname{div}\dot{u}^k\right|^2\right)\\
&\quad\leqslant C\sup_{0\leqslant s\leqslant T}\left[\Phi_K(s)\right]^2\exp\left\{C\int_0^t\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\}+\int_\Omega|g|^2\\
&\qquad+C\sup_{0\leqslant s\leqslant T}\Phi_K(s)\int_0^t\Phi_K(\tau)\left(1+\left\|\sqrt{m^k}\dot{u}^k\right\|_{L^2}\right)\left\|\nabla u^k\right\|_{L^6}^2\exp\left\{C\int_0^\tau\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\}\\
&\qquad+C\sup_{0\leqslant s\leqslant T}\Phi_K(s)\int_0^t\Phi_K(\tau)\left(1+\left\|\sqrt{m^{k-1}}\dot{u}^{k-1}\right\|_{L^2}\right)\left\|\nabla u^{k-1}\right\|_{L^6}^2\exp\left\{C\int_0^\tau\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}\,ds\right\},
\end{aligned}
$$

where $\dot{u}^k = u^k_t + u^{k-1}\cdot\nabla u^k$, $0 \leqslant t \leqslant T$*, and* C *is a positive constant, independent of* K, δ *and* T.

Proof. $(4.1)_3$ can be rewritten as

$$
m^k\dot{u}^k+\nabla P^k-Lu^k=0. \qquad (4.19)
$$

Differentiating (4.19) with respect to t, and using $(4.1)_1$, we conclude

$$\begin{aligned}&m^k\dot{u}_t^k+m^ku^{k-1}\cdot\nabla\dot{u}^k+\nabla P_t^k+\operatorname{div}\big(\nabla P^k\otimes u^{k-1}\big)\\&\quad=L\dot{u}^k-L\big(u^{k-1}\cdot\nabla u^k\big)+\operatorname{div}\big(Lu^k\otimes u^{k-1}\big).\end{aligned}\tag{4.20}$$

Multiplying (4.20) by $\dot{u}^k$, integrating the resulting equation over Ω, and using integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega m^k\big|\dot{u}^k\big|^2+\int_\Omega\big(\mu\big|\nabla\dot{u}^k\big|^2+(\mu+\lambda)\big|\operatorname{div}\dot{u}^k\big|^2\big)\\&=\int_\Omega\big(P_t^k\operatorname{div}\dot{u}^k+\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\cdot\nabla P^k\big)+\mu\int_\Omega\nabla\big(u^{k-1}\cdot\nabla u^k\big):\nabla\dot{u}^k+(\mu+\lambda)\int_\Omega\operatorname{div}\big(u^{k-1}\cdot\nabla u^k\big)\operatorname{div}\dot{u}^k\\&\quad-\int_\Omega\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\cdot\big(\mu\Delta u^k+(\mu+\lambda)\nabla\operatorname{div}u^k\big)\\&=\int_\Omega\big(P_t^k\operatorname{div}\dot{u}^k+\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\cdot\nabla P^k\big)+\mu\int_\Omega\big[\nabla\big(u^{k-1}\cdot\nabla u^k\big):\nabla\dot{u}^k-\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\cdot\Delta u^k\big]\\&\quad+(\mu+\lambda)\int_\Omega\big[\operatorname{div}\big(u^{k-1}\cdot\nabla u^k\big)\operatorname{div}\dot{u}^k-\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\cdot\nabla\operatorname{div}u^k\big]\\&=I_1+I_2+I_3.\end{aligned}\tag{4.21}$$

Now we estimate I_1, I_2 and I_3 as follows:

$$\begin{aligned}I_1&=\int_\Omega\big[\big(P_{m^k}^km_t^k+P_{n^k}^kn_t^k\big)\operatorname{div}\dot{u}^k+\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\cdot\nabla P^k\big]\,dx\\&=\int_\Omega\big[\big(-m^kP_{m^k}^k-n^kP_{n^k}^k\big)\operatorname{div}u^{k-1}\operatorname{div}\dot{u}^k-u^{k-1}\cdot\nabla P^k\operatorname{div}\dot{u}^k+\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\cdot\nabla P^k\big]\\&=\int_\Omega\big[\big(-m^kP_{m^k}^k-n^kP_{n^k}^k\big)\operatorname{div}u^{k-1}\operatorname{div}\dot{u}^k+P^k\operatorname{div}\big(u^{k-1}\operatorname{div}\dot{u}^k\big)-P^k\operatorname{div}\big(u^{k-1}\cdot\nabla\dot{u}^k\big)\big]\\&=\int_\Omega\big[\big(-m^kP_{m^k}^k-n^kP_{n^k}^k\big)\operatorname{div}u^{k-1}\operatorname{div}\dot{u}^k+P^k\big(\operatorname{div}u^{k-1}\operatorname{div}\dot{u}^k-\big(\nabla u^{k-1}\big)':\nabla\dot{u}^k\big)\big]\\&\leqslant C\Phi_K(t)\big\|\nabla u^{k-1}\big\|_{L^2}\big\|\nabla\dot{u}^k\big\|_{L^2},\end{aligned}\tag{4.22}$$

where we have used integration by parts, $(4.1)_1$, $(4.1)_2$, Lemma 4.2, Lemma 4.3 and Hölder inequality.

$$\begin{aligned}I_2&=\mu\int_\Omega\big[\nabla\big(u^{k-1}\cdot\nabla u^k\big):\nabla\dot{u}^k+\nabla\big(u^{k-1}\cdot\nabla\dot{u}^k\big):\nabla u^k\big]\\&=\mu\int_\Omega\big[\nabla\big(u^{k-1}\cdot\nabla u^k\big):\nabla\dot{u}^k+\big(\nabla u^{k-1}\cdot\nabla\dot{u}^k\big):\nabla u^k+\big(u^{k-1}\cdot\nabla\big)\nabla\dot{u}^k:\nabla u^k\big]\\&=\mu\int_\Omega\big[\nabla\big(u^{k-1}\cdot\nabla u^k\big):\nabla\dot{u}^k+\big(\nabla u^{k-1}\cdot\nabla\dot{u}^k\big):\nabla u^k-\operatorname{div}u^{k-1}\nabla\dot{u}^k:\nabla u^k-\big(u^{k-1}\cdot\nabla\big)\nabla u^k:\nabla\dot{u}^k\big]\\&=\mu\int_\Omega\big[\big(\nabla u^{k-1}\cdot\nabla u^k\big):\nabla\dot{u}^k+\big(\nabla u^{k-1}\cdot\nabla\dot{u}^k\big):\nabla u^k-\operatorname{div}u^{k-1}\nabla\dot{u}^k:\nabla u^k\big]\\&\leqslant C\big\|\nabla\dot{u}^k\big\|_{L^2}\big\|\nabla u^{k-1}\big\|_{L^4}\big\|\nabla u^k\big\|_{L^4},\end{aligned}\tag{4.23}$$

where we have used integration by parts and Hölder inequality.

$$
\begin{aligned}
I_3 &= (\mu+\lambda)\int_\Omega \big[\operatorname{div}\big(u^{k-1}\cdot\nabla u^k\big)\operatorname{div}\dot u^k + \nabla\cdot\big(u^{k-1}\cdot\nabla\dot u^k\big)\operatorname{div}u^k\big] \\
&= (\mu+\lambda)\int_\Omega \big[\operatorname{div}\big(u^{k-1}\cdot\nabla u^k\big)\operatorname{div}\dot u^k + \operatorname{div}u^k\big(\nabla u^{k-1}\big)' : \nabla\dot u^k + \big(u^{k-1}\cdot\nabla\operatorname{div}\dot u^k\big)\operatorname{div}u^k\big] \\
&= (\mu+\lambda)\int_\Omega \big[\operatorname{div}\big(u^{k-1}\cdot\nabla u^k\big)\operatorname{div}\dot u^k + \operatorname{div}u^k\big(\nabla u^{k-1}\big)' : \nabla\dot u^k - \operatorname{div}u^{k-1}\operatorname{div}\dot u^k\operatorname{div}u^k \\
&\qquad - \big(u^{k-1}\cdot\nabla\operatorname{div}u^k\big)\operatorname{div}\dot u^k\big] \\
&= (\mu+\lambda)\int_\Omega \big[\big(\nabla u^{k-1}\big)' : \nabla u^k\operatorname{div}\dot u^k + \operatorname{div}u^k\big(\nabla u^{k-1}\big)' : \nabla\dot u^k - \operatorname{div}u^{k-1}\operatorname{div}\dot u^k\operatorname{div}u^k\big] \\
&\leqslant C\big\|\nabla\dot u^k\big\|_{L^2}\big\|\nabla u^{k-1}\big\|_{L^4}\big\|\nabla u^k\big\|_{L^4}.
\end{aligned}
\tag{4.24}
$$

Substituting (4.22)–(4.24) into (4.21), and using Cauchy inequality and Corollary 4.5, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega m^k\big|\dot u^k\big|^2 + \int_\Omega \big(\mu\big|\nabla\dot u^k\big|^2 + (\mu+\lambda)\big|\operatorname{div}\dot u^k\big|^2\big) \\
&\quad\leqslant C\big[\Phi_K(t)\big]^2\big\|\nabla u^{k-1}\big\|_{L^2}^2 + C\big\|\nabla u^{k-1}\big\|_{L^4}^4 + C\big\|\nabla u^k\big\|_{L^4}^4 \\
&\quad\leqslant C\big[\Phi_K(t)\big]^2\sup_{0\leqslant s\leqslant T}\big[\Phi_K(s)\big]^2\exp\left\{C\int_0^t\big[\Phi_K(s)\big]^{\frac{r+1}{r-3}}\big[\Psi_{K,r}(s)\big]^{\frac{2}{r-3}}\,ds\right\} \\
&\qquad + C\big\|\nabla u^{k-1}\big\|_{L^4}^4 + C\big\|\nabla u^k\big\|_{L^4}^4.
\end{aligned}
\tag{4.25}
$$

In the following, we estimate the term $\|\nabla u^k\|_{L^4}^4$. From Eq. $(4.1)_3$ and (4.10), we know that w^k satisfies

$$
\begin{cases}
Lw^k = m^k\dot u^k, & \text{in } \Omega,\\
w^k(x) = 0, & \text{on } \partial\Omega.
\end{cases}
\tag{4.26}
$$

By (4.26) and Lemma 2.3, we get

$$
\big\|w^k\big\|_{H^2}\leqslant C\big\|m^k\dot u^k\big\|_{L^2}\leqslant C\sqrt{\Phi_K(t)}\big\|\sqrt{m^k}\dot u^k\big\|_{L^2},
$$

which together with the interpolation inequality, Sobolev inequality, and $(4.11)_1$ yields

$$
\begin{aligned}
\big\|\nabla u^k\big\|_{L^4}^4 &\leqslant C\big\|\nabla u^k\big\|_{L^2}\big\|\nabla u^k\big\|_{L^6}^3 \\
&\leqslant C\big\|\nabla u^k\big\|_{L^2}\big\|\nabla u^k\big\|_{L^6}^2\big(\big\|\nabla w^k\big\|_{L^6} + \big\|\nabla h^k\big\|_{L^6}\big) \\
&\leqslant C\big\|\nabla u^k\big\|_{L^2}\big\|\nabla u^k\big\|_{L^6}^2\big[\Phi_K(t) + \big\|\nabla w^k\big\|_{H^1}\big] \\
&\leqslant C\big\|\nabla u^k\big\|_{L^2}\big\|\nabla u^k\big\|_{L^6}^2\big[\Phi_K(t) + \sqrt{\Phi_K(t)}\big\|\sqrt{m^k}\dot u^k\big\|_{L^2}\big] \\
&\leqslant C\Phi_K(t)\big\|\nabla u^k\big\|_{L^2}\big\|\nabla u^k\big\|_{L^6}^2\big(1 + \big\|\sqrt{m^k}\dot u^k\big\|_{L^2}\big).
\end{aligned}
$$

This together with Corollary 4.5 gives

$$
\big\|\nabla u^k\big\|_{L^4}^4\leqslant C\big(1+\big\|\sqrt{m^k}\dot u^k\big\|_{L^2}\big)\Phi_K(t)\big\|\nabla u^k\big\|_{L^6}^2\sup_{0\leqslant s\leqslant T}\Phi_K(s)\exp\left\{C\int_0^t\big[\Phi_K(s)\big]^{\frac{r+1}{r-3}}\big[\Psi_{K,r}(s)\big]^{\frac{2}{r-3}}\,ds\right\}. \tag{4.27}
$$

Similarly, we have

$$
\big\|\nabla u^{k-1}\big\|_{L^4}^4\leqslant C\big(1+\big\|\sqrt{m^{k-1}}\dot u^{k-1}\big\|_{L^2}\big)\Phi_K(t)\big\|\nabla u^{k-1}\big\|_{L^6}^2\sup_{0\leqslant s\leqslant T}\Phi_K(s)\exp\left\{C\int_0^t\big[\Phi_K(s)\big]^{\frac{r+1}{r-3}}\big[\Psi_{K,r}(s)\big]^{\frac{2}{r-3}}\,ds\right\}. \tag{4.28}
$$

Substituting (4.27) and (4.28) into (4.25), we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega m^k|\dot u^k|^2+\int_\Omega\left(\mu|\nabla\dot u^k|^2+(\mu+\lambda)|\operatorname{div}\dot u^k|^2\right)\\
&\quad\leqslant C\Phi_K(t)\left(1+\left\|\sqrt{m^k}\dot u^k\right\|_{L^2}\right)\left\|\nabla u^k\right\|_{L^6}^2\sup_{0\leqslant s\leqslant T}\Phi_K(s)\exp\left\{C\int_0^t\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}ds\right\}\\
&\qquad+C\Phi_K(t)\left(1+\left\|\sqrt{m^{k-1}}\dot u^{k-1}\right\|_{L^2}\right)\left\|\nabla u^{k-1}\right\|_{L^6}^2\sup_{0\leqslant s\leqslant T}\Phi_K(s)\exp\left\{C\int_0^t\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}ds\right\}\\
&\qquad+C\left[\Phi_K(t)\right]^2\sup_{0\leqslant s\leqslant T}\left[\Phi_K(s)\right]^2\exp\left\{C\int_0^t\left[\Phi_K(s)\right]^{\frac{r+1}{r-3}}\left[\Psi_{K,r}(s)\right]^{\frac{2}{r-3}}ds\right\}.
\end{aligned}$$

Integrating over $(0,t)$ and using (1.6), we complete the proof of Lemma 4.6. □

Note that $T>0$ and $r\in(3,4]$ are arbitrary for all the above estimates which will be useful to get the blow-up criterion of the solution in the next section. To obtain the strong solutions, we have to take T small enough. Therefore, we assume $T\in(0,1)$. Moreover, we take $r=4$ for simplicity.

Suppose for $1\leqslant k\leqslant K$

$$\left\|u^{k-1}\right\|_{L^2(0,T;W^{2,q})}+\left\|u^{k-1}\right\|_{L^\infty(0,T;H^2)}+\sup_{0\leqslant t\leqslant T}\int_\Omega m^{k-1}\left|\dot u^{k-1}\right|^2\leqslant M_1, \tag{4.29}$$

for $M_1>1$ large enough.

Throughout the rest of the section, we denote by C a generic positive constant which may be dependent on μ, λ, Ω, m_0, n_0, u_0 and other known constants but independent of M_1, K, δ and T.

Lemma 4.7. *Under the conditions of Theorem* 4.1*, we have for any* $k\geqslant1$ *and* $T\leqslant M_1^{-8}$

$$\begin{cases}
\sup\limits_{0\leqslant t\leqslant T}\int_\Omega m^k\left|u_t^k\right|^2+\int_0^T\int_\Omega\left|\nabla u_t^k\right|^2\leqslant CM_1^4,\\
\left\|u^k\right\|_{L^2(0,T;W^{2,q})}+\left\|u^k\right\|_{L^\infty(0,T;H_0^1\cap H^2)}\leqslant M_1,\\
\left\|m^k\right\|_{L^\infty(0,T;W^{1,q})}+\left\|n^k\right\|_{L^\infty(0,T;W^{1,q})}\leqslant C,\\
\dfrac{\underline{s}_0\delta}{C}\leqslant\underline{s}_0m^k\leqslant n^k\leqslant\bar s_0m^k,\quad\text{in } Q_T,\\
\left\|P^k\right\|_{L^\infty(Q_T)}+\left\|P^k_{m^k}\right\|_{L^\infty(Q_T)}+\left\|P^k_{n^k}\right\|_{L^\infty(Q_T)}\leqslant C.
\end{cases}$$

Proof. By (4.29), (3.4), $(4.1)_1$–$(4.1)_2$ and Sobolev inequality, we have

$$\begin{cases}
\sup\limits_{0\leqslant t\leqslant T}\left\|m^k(t)\right\|_{W^{1,q}}\leqslant C\exp\left\{CM_1T^{\frac12}\right\},\\
\sup\limits_{0\leqslant t\leqslant T}\left\|n^k(t)\right\|_{W^{1,q}}\leqslant C\exp\left\{CM_1T^{\frac12}\right\}.
\end{cases}$$

Denote $T_1=M_1^{-2}$, we get for $T\leqslant T_1$

$$\begin{cases}
\sup\limits_{0\leqslant t\leqslant T}\left\|m^k(t)\right\|_{W^{1,q}}\leqslant C,\\
\sup\limits_{0\leqslant t\leqslant T}\left\|n^k(t)\right\|_{W^{1,q}}\leqslant C.
\end{cases} \tag{4.30}$$

$(4.30)_1$ and Sobolev inequality give

$$\sup_{0\leqslant t\leqslant T} \Phi_K(t) \leqslant C. \tag{4.31}$$

This together with (4.5) implies

$$\|P^k\|_{L^\infty(Q_T)} \leqslant C. \tag{4.32}$$

By (4.6), (4.7), (4.30) and (4.32), we have

$$\|P^k\|_{L^\infty(0,T;W^{1,q})} \leqslant C. \tag{4.33}$$

We obtain from (4.29), (4.31) and Sobolev inequality

$$\sup_{0\leqslant t\leqslant T} \Psi_{K,4}(t) \leqslant CM_1^4. \tag{4.34}$$

It follows from Corollary 4.5, (4.31), and (4.34) that for $T \leqslant T_1$

$$\|\nabla u^k\|_{L^2(0,T;L^6)} + \|\nabla u^k\|_{L^\infty(0,T;L^2)} \leqslant C\exp\{CTM_1^8\}.$$

Take $T_2 = M_1^{-8}$, we have for $T \leqslant T_2$

$$\|\nabla u^k\|_{L^2(0,T;L^6)} + \|\nabla u^k\|_{L^\infty(0,T;L^2)} \leqslant C. \tag{4.35}$$

By Lemma 4.6, (4.29), (4.31), (4.34), (4.35), and Sobolev inequality, we get for $0 \leqslant t \leqslant T \leqslant T_2$

$$\begin{aligned}
&\int_\Omega m^k|\dot u^k|^2 + \int_0^t\int_\Omega \left(\mu|\nabla \dot u^k|^2 + (\mu+\lambda)|\operatorname{div}\dot u^k|^2\right) \\
&\quad\leqslant C + \int_\Omega |g|^2 + C\int_0^t \left(1 + \|\sqrt{m^k}\dot u^k\|_{L^2}\right)\|\nabla u^k\|_{L^6}^2 + C\int_0^t \left(1 + \|\sqrt{m^{k-1}}\dot u^{k-1}\|_{L^2}\right)\|\nabla u^{k-1}\|_{H^1}^2 \\
&\quad\leqslant C + \int_\Omega |g|^2 + C\Big(1 + \sup_{0\leqslant t\leqslant T}\|\sqrt{m^k}\dot u^k\|_{L^2}\Big) + CM_1^{\frac52}T.
\end{aligned}$$

Using Cauchy inequality, we have for $T \leqslant T_2$

$$\sup_{0\leqslant t\leqslant T}\int_\Omega m^k|\dot u^k|^2 + \int_0^T\int_\Omega |\nabla \dot u^k|^2 \leqslant C + C\int_\Omega |g|^2 + CM_1^{\frac52}T. \tag{4.36}$$

By (4.19), Lemma 2.3, (4.31), (4.33) and (4.36), we get for $T \leqslant T_2$

$$\|u^k\|_{L^2(0,T;W^{2,q})} + \|u^k\|_{L^\infty(0,T;H^2)} \leqslant C + C\int_\Omega |g|^2 + CM_1^{\frac52}T.$$

Since $T \leqslant M_1^{-8}$, we have

$$\sup_{0\leqslant t\leqslant T}\int_\Omega m^k|\dot u^k|^2 + \int_0^T\int_\Omega |\nabla \dot u^k|^2 + \|u^k\|_{L^2(0,T;W^{2,q})} + \|u^k\|_{L^\infty(0,T;H^2)} \leqslant C + C\int_\Omega |g|^2.$$

Let $M_1 \geqslant C + C\int_\Omega |g|^2$, we obtain for $T \leqslant T_2$

$$\sup_{0\leqslant t\leqslant T}\int_\Omega m^k|\dot u^k|^2 + \int_0^T\int_\Omega |\nabla \dot u^k|^2 + \|u^k\|_{L^2(0,T;W^{2,q})} + \|u^k\|_{L^\infty(0,T;H^2)} \leqslant M_1. \tag{4.37}$$

By induction, (4.37) is valid for any $k \in [1, K]$. Since M_1 is independent of K, and K is arbitrary, we conclude that (4.37) is actually valid for all $k \geqslant 1$.

From (4.37), we obtain for $T \leqslant T_2$

$$\sup_{0\leqslant t\leqslant T}\int_\Omega m^k|u_t^k|^2 + \int_0^T\int_\Omega|\nabla u_t^k|^2 \leqslant CM_1^4.$$

Summarily, we have for any $k \geqslant 1$ and $T \leqslant T_2$

$$\begin{cases} \sup\limits_{0\leqslant t\leqslant T}\int_\Omega m^k|u_t^k|^2 + \int_0^T\int_\Omega|\nabla u_t^k|^2 \leqslant CM_1^4, \\ \|u^k\|_{L^2(0,T;W^{2,q})} + \|u^k\|_{L^\infty(0,T;H_0^1\cap H^2)} \leqslant M_1, \\ \|m^k\|_{L^\infty(0,T;W^{1,q})} + \|n^k\|_{L^\infty(0,T;W^{1,q})} \leqslant C, \\ \dfrac{\underline{s}_0\delta}{C} \leqslant \underline{s}_0 m^k \leqslant n^k \leqslant \bar{s}_0 m^k, \quad \text{in } Q_T, \\ \|P^k\|_{L^\infty(Q_T)} + \|P_{m^k}^k\|_{L^\infty(Q_T)} + \|P_{n^k}^k\|_{L^\infty(Q_T)} \leqslant C. \end{cases} \tag{4.38}$$

The proof of Lemma 4.7 is completed. □

We will show that the full sequence (m^k, n^k, u^k) converges to a solution of (1.1)–(1.5). To do this, we denote

$$\overline{m}^{k+1} = m^{k+1} - m^k, \qquad \overline{n}^{k+1} = n^{k+1} - n^k, \qquad \overline{u}^{k+1} = u^{k+1} - u^k \quad \text{and} \quad \overline{P}^{k+1} = P^{k+1} - P^k.$$

It follows from $(4.1)_3$ that

$$m^{k+1}\overline{u}_t^{k+1} + m^{k+1}u^k\cdot\nabla\overline{u}^{k+1} - L\overline{u}^{k+1} + \nabla\overline{P}^{k+1} = \overline{m}^{k+1}\big(-u_t^k - u^k\cdot\nabla u^k\big) - m^k\overline{u}^k\cdot\nabla u^k. \tag{4.39}$$

Multiplying (4.39) by $\overline{u}^{k+1}$, and using (4.38) and Sobolev inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega m^{k+1}|\overline{u}^{k+1}|^2 + \int_\Omega\big[\mu|\nabla\overline{u}^{k+1}|^2 + (\mu+\lambda)|\operatorname{div}\overline{u}^{k+1}|^2\big] \\ &= \int_\Omega \overline{P}^{k+1}\operatorname{div}\overline{u}^{k+1} + \int_\Omega \overline{m}^{k+1}\big(-u_t^k - u^k\cdot\nabla u^k\big)\overline{u}^{k+1} - \int_\Omega m^k\big(\overline{u}^k\cdot\nabla u^k\big)\cdot\overline{u}^{k+1} \\ &\leqslant \|\overline{P}^{k+1}\|_{L^2}\|\operatorname{div}\overline{u}^{k+1}\|_{L^2} + \|\overline{m}^{k+1}\|_{L^2}\|u_t^k + u^k\cdot\nabla u^k\|_{L^3}\|\overline{u}^{k+1}\|_{L^6} \\ &\quad + \|\sqrt{m^k}\,\overline{u}^k\|_{L^2}\|\sqrt{m^k}\|_{L^\infty}\|\overline{u}^{k+1}\|_{L^6}\|\nabla u^k\|_{L^3} \\ &\leqslant \|\overline{P}^{k+1}\|_{L^2}\|\operatorname{div}\overline{u}^{k+1}\|_{L^2} + C\|\overline{m}^{k+1}\|_{L^2}\big(\|\nabla u_t^k\|_{L^2} + M_1^2\big)\|\nabla\overline{u}^{k+1}\|_{L^2} + C\|\sqrt{m^k}\,\overline{u}^k\|_{L^2}\|\nabla\overline{u}^{k+1}\|_{L^2}M_1 \\ &\leqslant \frac{1}{2}\int_\Omega\big[\mu|\nabla\overline{u}^{k+1}|^2 + (\mu+\lambda)|\operatorname{div}\overline{u}^{k+1}|^2\big] + C\|\overline{n}^{k+1}\|_{L^2}^2 + C\|\overline{m}^{k+1}\|_{L^2}^2\big(\|\nabla u_t^k\|_{L^2}^2 + M_1^4\big) \\ &\quad + CM_1^2\|\sqrt{m^k}\,\overline{u}^k\|_{L^2}^2. \end{aligned}$$

This gives

$$\begin{aligned} &\frac{d}{dt}\int_\Omega m^{k+1}|\overline{u}^{k+1}|^2 + \int_\Omega\big[\mu|\nabla\overline{u}^{k+1}|^2 + (\mu+\lambda)|\operatorname{div}\overline{u}^{k+1}|^2\big] \\ &\quad \leqslant C\|\overline{n}^{k+1}\|_{L^2}^2 + C\|\overline{m}^{k+1}\|_{L^2}^2\big(\|\nabla u_t^k\|_{L^2}^2 + M_1^4\big) + CM_1^2\|\sqrt{m^k}\,\overline{u}^k\|_{L^2}^2. \end{aligned} \tag{4.40}$$

From $(4.1)_1$, we have

$$\overline{m}_t^{k+1} + \overline{m}^{k+1}\operatorname{div}u^k + m^k\operatorname{div}\overline{u}^k + u^k\cdot\nabla\overline{m}^{k+1} + \overline{u}^k\cdot\nabla m^k = 0. \tag{4.41}$$

Multiplying (4.41) by $\overline{m}^{k+1}$, integrating over Ω, and using integration by parts and (4.38), we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left|\overline{m}^{k+1}\right|^2 &= -\int_\Omega \left|\overline{m}^{k+1}\right|^2 \operatorname{div} u^k - 2\int_\Omega \left(m^k \operatorname{div}\bar{u}^k + \bar{u}^k\cdot\nabla m^k\right)\overline{m}^{k+1} \\
&\leqslant C\left\|u^k\right\|_{W^{2,q}}\left\|\overline{m}^{k+1}\right\|_{L^2}^2 + C\left\|\overline{m}^{k+1}\right\|_{L^2}\left(\left\|\nabla\bar{u}^k\right\|_{L^2} + \left\|\bar{u}^k\cdot\nabla m^k\right\|_{L^2}\right) \\
&\leqslant C\left\|u^k\right\|_{W^{2,q}}\left\|\overline{m}^{k+1}\right\|_{L^2}^2 + C\left\|\overline{m}^{k+1}\right\|_{L^2}\left(\left\|\nabla\bar{u}^k\right\|_{L^2} + \left\|\nabla m^k\right\|_{L^3}\left\|\bar{u}^k\right\|_{L^6}\right) \\
&\leqslant C\left\|u^k\right\|_{W^{2,q}}\left\|\overline{m}^{k+1}\right\|_{L^2}^2 + C\left\|\overline{m}^{k+1}\right\|_{L^2}\left\|\nabla\bar{u}^k\right\|_{L^2}.
\end{aligned} \tag{4.42}$$

Similarly, we have from $(4.1)_2$

$$\frac{d}{dt}\int_\Omega \left|\bar{n}^{k+1}\right|^2 \leqslant C\left\|u^k\right\|_{W^{2,q}}\left\|\bar{n}^{k+1}\right\|_{L^2}^2 + C\left\|\bar{n}^{k+1}\right\|_{L^2}\left\|\nabla\bar{u}^k\right\|_{L^2}. \tag{4.43}$$

By (4.42)–(4.43) and Cauchy inequality, we have

$$\frac{d}{dt}\int_\Omega \left(\left|\overline{m}^{k+1}\right|^2 + \left|\bar{n}^{k+1}\right|^2\right) \leqslant C\left\|u^k\right\|_{W^{2,q}}\left(\left\|\overline{m}^{k+1}\right\|_{L^2}^2 + \left\|\bar{n}^{k+1}\right\|_{L^2}^2\right) + C\left(\left\|\overline{m}^{k+1}\right\|_{L^2} + \left\|\bar{n}^{k+1}\right\|_{L^2}\right)\left\|\nabla\bar{u}^k\right\|_{L^2}. \tag{4.44}$$

By (4.40), (4.44) and Cauchy inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\left|\overline{m}^{k+1}\right|^2 + \left|\bar{n}^{k+1}\right|^2 + m^{k+1}\left|\bar{u}^{k+1}\right|^2\right) + \int_\Omega \left[\mu\left|\nabla\bar{u}^{k+1}\right|^2 + (\mu+\lambda)\left|\operatorname{div}\bar{u}^{k+1}\right|^2\right] \\
&\leqslant C\left(\left\|u^k\right\|_{W^{2,q}} + \left\|\nabla u_t^k\right\|_{L^2}^2 + M_1^4\right)\left(\left\|\overline{m}^{k+1}\right\|_{L^2}^2 + \left\|\bar{n}^{k+1}\right\|_{L^2}^2\right) + C\left(\left\|\overline{m}^{k+1}\right\|_{L^2} + \left\|\bar{n}^{k+1}\right\|_{L^2}\right)\left\|\nabla\bar{u}^k\right\|_{L^2} \\
&\quad + CM_1^2\left\|\sqrt{m^k}\bar{u}^k\right\|_{L^2}^2 \\
&\leqslant C\left(\left\|u^k\right\|_{W^{2,q}} + \left\|\nabla u_t^k\right\|_{L^2}^2 + M_1^4 + \frac{1}{\varepsilon}\right)\left(\left\|\overline{m}^{k+1}\right\|_{L^2}^2 + \left\|\bar{n}^{k+1}\right\|_{L^2}^2\right) + \varepsilon\int_\Omega \mu\left|\nabla\bar{u}^k\right|^2 + CM_1^2\left\|\sqrt{m^k}\bar{u}^k\right\|_{L^2}^2,
\end{aligned}$$

for any $\varepsilon > 0$. Denote

$$\begin{aligned}
\varphi^{k+1}(t) &= \int_\Omega \left(\left|\overline{m}^{k+1}\right|^2 + \left|\bar{n}^{k+1}\right|^2 + m^{k+1}\left|\bar{u}^{k+1}\right|^2\right)(t), \\
\psi^{k+1}(t) &= \int_\Omega \left[\mu\left|\nabla\bar{u}^{k+1}\right|^2 + (\mu+\lambda)\left|\operatorname{div}\bar{u}^{k+1}\right|^2\right](t), \\
D^k(\varepsilon,t) &= C\left(\left\|u^k\right\|_{W^{2,q}} + \left\|\nabla u_t^k\right\|_{L^2}^2 + M_1^4 + \frac{1}{\varepsilon}\right).
\end{aligned}$$

We have

$$\frac{d}{dt}\varphi^{k+1}(t) + \psi^{k+1}(t) \leqslant D^k(\varepsilon,t)\varphi^{k+1}(t) + \varepsilon\psi^k(t) + CM_1^2\varphi^k(t).$$

This together with Gronwall inequality, (4.38) and $\varphi^{k+1}(0) = 0$ implies

$$\varphi^{k+1}(t) + \int_0^t \psi^{k+1}(s) \leqslant \exp\left(\widetilde{C} + \frac{CT}{\varepsilon}\right)\int_0^t \left[\varepsilon\psi^k(s) + CM_1^2\varphi^k(s)\right], \tag{4.45}$$

where $\widetilde{C}$ depends on M_1 and other known constants related to C.

By (4.45), we have

$$\sup_{0\leqslant t\leqslant T}\varphi^{k+1}(t)+\int_0^T\psi^{k+1}(s)\leqslant\exp\biggl(\widetilde{C}+\frac{CT}{\varepsilon}\biggr)\bigl(CM_1^2T+\varepsilon\bigr)\biggl(\sup_{0\leqslant t\leqslant T}\varphi^{k}(t)+\int_0^T\psi^{k}(s)\biggr).$$

Take $\varepsilon=\frac{1}{16}\exp(\widetilde{C}+C)(CM_1^2+1)$, and $T_0=\min\{T_2,\varepsilon\}$, we have for $T=T_0$

$$\sup_{0\leqslant t\leqslant T_0}\bigl[\varphi^{k+1}(t)\bigr]^{\frac{1}{2}}+\biggl(\int_0^{T_0}\psi^{k+1}(s)\biggr)^{\frac{1}{2}}\leqslant\frac{1}{2}\biggl\{\sup_{0\leqslant t\leqslant T_0}\bigl(\varphi^{k}(t)\bigr)^{\frac{1}{2}}+\biggl(\int_0^{T_0}\psi^{k}(s)\biggr)^{\frac{1}{2}}\biggr\}.$$

This implies

$$\sum_{k=1}^{\infty}\biggl\{\sup_{0\leqslant t\leqslant T_0}\bigl[\varphi^{k+1}(t)\bigr]^{\frac{1}{2}}+\biggl(\int_0^{T_0}\psi^{k+1}(s)\biggr)^{\frac{1}{2}}\biggr\}<\infty.$$

Recalling the notations of $\varphi^{k+1}(t)$ and $\psi^{k+1}(t)$, we get

$$\sum_{k=1}^{\infty}\bigl(\|\overline{m}^{k+1}\|_{L^\infty(0,T_0;L^2)}+\|\bar{n}^{k+1}\|_{L^\infty(0,T_0;L^2)}+\|\bar{u}^{k+1}\|_{L^2(0,T_0;H_0^1)}\bigr)<\infty. \tag{4.46}$$

Denote $m=\sum_{i=2}^{\infty}\overline{m}^i+m^1$, $n=\sum_{i=2}^{\infty}\bar{n}^i+n^1$, $u=\sum_{i=1}^{\infty}\bar{u}^i$, we have

$$\begin{aligned}
&\|m^k-m\|_{L^\infty(0,T_0;L^2)}+\|n^k-n\|_{L^\infty(0,T_0;L^2)}+\|u^k-u\|_{L^2(0,T_0;H_0^1)}\\
&\quad=\Biggl\|\sum_{i=k+1}^{\infty}\overline{m}^i\Biggr\|_{L^\infty(0,T_0;L^2)}+\Biggl\|\sum_{i=k+1}^{\infty}\bar{n}^i\Biggr\|_{L^\infty(0,T_0;L^2)}+\Biggl\|\sum_{i=k+1}^{\infty}\bar{u}^i\Biggr\|_{L^2(0,T_0;H_0^1)}\\
&\quad\leqslant\sum_{i=k+1}^{\infty}\bigl(\|\overline{m}^i\|_{L^\infty(0,T_0;L^2)}+\|\bar{n}^i\|_{L^\infty(0,T_0;L^2)}+\|\bar{u}^i\|_{L^2(0,T_0;H_0^1)}\bigr)\to 0,
\end{aligned}$$

as $k\to\infty$. Here we have used (4.46).

Therefore, we conclude the convergence of the full sequence (m^k,n^k,u^k) as $k\to\infty$

$$\begin{cases}
m^k\to m, & \text{in } L^\infty\bigl(0,T_0;L^2\bigr),\\
n^k\to n, & \text{in } L^\infty\bigl(0,T_0;L^2\bigr),\\
u^k\to u, & \text{in } L^2\bigl(0,T_0;H_0^1\bigr).
\end{cases} \tag{4.47}$$

It follows from Lemma 4.7, (4.47) and lower semi-continuity of norms, we conclude that (m,n,u) is a solution to (1.1)–(1.5) under the conditions of Theorem 4.1, and that the following estimates uniform for δ are obtained

$$\begin{cases}
\displaystyle\sup_{0\leqslant t\leqslant T_0}\int_\Omega m|u_t|^2+\int_0^{T_0}\int_\Omega|\nabla u_t|^2\leqslant\widetilde{C},\\
\|u\|_{L^2(0,T_0;W^{2,q})}+\|u\|_{L^\infty(0,T_0;H_0^1\cap H^2)}+\|m\|_{L^\infty(0,T_0;W^{1,q})}+\|n\|_{L^\infty(0,T_0;W^{1,q})}\leqslant\widetilde{C},\\
\dfrac{\underline{s}_0\delta}{\widetilde{C}}\leqslant\underline{s}_0m\leqslant n\leqslant\bar{s}_0m, \quad\text{in } Q_{T_0},\\
\|P\|_{L^\infty(Q_{T_0})}+\|P_m\|_{L^\infty(Q_{T_0})}+\|P_n\|_{L^\infty(Q_{T_0})}\leqslant\widetilde{C}.
\end{cases}$$

The uniqueness can be obtained similar to the proceeding of the convergence of full sequence. The proof of Theorem 4.1 is completed. □

Proof of Theorem 1.1. Denote $m_0^\delta=m_0+\delta$, $n_0^\delta=n_0+\delta$, we have as $\delta\to 0$

$$\begin{cases} m_0^\delta \to m_0, & \text{in } W^{1,q}, \\ n_0^\delta \to n_0, & \text{in } W^{1,q}. \end{cases} \tag{4.48}$$

Since $0 \leqslant \underline{s}_0 m_0 \leqslant n_0 \leqslant \bar{s}_0 m_0$, in $\overline{\Omega}$, without loss of generality, we assume $0 < \underline{s}_0 \leqslant 1 \leqslant \bar{s}_0$, we have

$$\underline{s}_0 m_0^\delta \leqslant n_0^\delta \leqslant \bar{s}_0 m_0^\delta, \quad \text{in } \overline{\Omega}. \tag{4.49}$$

By Lemma 2.3, we can find a $u_0^\delta \in H_0^1 \cap H^2$ for each $\delta > 0$ such that

$$\begin{cases} -Lu_0^\delta + \nabla P\big(m_0^\delta, n_0^\delta\big) = \sqrt{m_0^\delta} g, & \text{in } \Omega, \\ u_0^\delta|_{\partial\Omega} = 0. \end{cases} \tag{4.50}$$

It follows from (1.6), (4.48), (4.50), and Lemma 2.3 that

$$u_0^\delta \to u_0, \quad \text{in } H^2,$$

as $\delta \to 0$.

Consider (1.1)–(1.5) with initial data (m_0, n_0, u_0) replaced by $(m_0^\delta, n_0^\delta, u_0^\delta)$, we obtain from Theorem 4.1 that there exists a $T_0 > 0$ independent of δ and a unique solution $(m^\delta, n^\delta, u^\delta)$ for each $\delta > 0$ with the following estimates:

$$\begin{cases} \sup\limits_{0 \leqslant t \leqslant T_0} \int_\Omega m^\delta \big|u_t^\delta\big|^2 + \int_0^{T_0} \int_\Omega \big|\nabla u_t^\delta\big|^2 \leqslant C, \\ \big\|u^\delta\big\|_{L^2(0,T_0;W^{2,q})} + \big\|u^\delta\big\|_{L^\infty(0,T_0;H_0^1\cap H^2)} + \big\|m^\delta\big\|_{L^\infty(0,T_0;W^{1,q})} + \big\|n^\delta\big\|_{L^\infty(0,T_0;W^{1,q})} \leqslant C, \\ \dfrac{\underline{s}_0\delta}{C} \leqslant \underline{s}_0 m^\delta \leqslant n^\delta \leqslant \bar{s}_0 m^\delta, \quad \text{in } Q_{T_0}, \\ \|P\|_{L^\infty(Q_{T_0})} + \|P_{m^\delta}\|_{L^\infty(Q_{T_0})} + \|P_{n^\delta}\|_{L^\infty(Q_{T_0})} \leqslant C, \end{cases}$$

where C is a positive constant, independent of δ.

Taking $\delta \to 0$ (take subsequence if necessary), and using the similar arguments in [1], under the conditions of Theorem 1.1, we get a solution (m, n, u) to (1.1)–(1.5) with the regularities like in Theorem 1.1. The uniqueness can be proved by the similar arguments in the proof of Theorem 4.1. The proof of Theorem 1.1 is completed. □

5. Proof of Theorem 1.2

Let (m, n, u) be a strong solution to the problem (1.1)–(1.5) in Q_T with the regularity stated in Theorem 1.1. We assume that the opposite holds, i.e.

$$\lim \sup_{T \to T^*} \|m\|_{L^\infty(0,T;L^\infty)} \leqslant M < \infty. \tag{5.1}$$

In this section, we denote by C a generic positive constant which may depend on μ, λ, Ω, m_0, n_0, u_0, M, T^*, and the parameters in the expression of P in (1.5).

Similar to Lemma 4.2, we get the first lemma:

Lemma 5.1. *Under the conditions of Theorem* 1.2*, we have for all* $0 \leqslant T < T^*$

$$\underline{s}_0 m \leqslant n \leqslant \bar{s}_0 m, \quad \text{in } Q_T.$$

Lemma 5.2. *Under the conditions of Theorem* 1.2*, there exists some* $r \in (3, 4]$ *such that*

$$\sup_{0 \leqslant t \leqslant T} \int_\Omega m|u|^r \, dx \leqslant C, \quad 0 \leqslant T < T^*.$$

Proof. Multiplying $(1.1)_3$ by $r|u|^{r-2}u$, integrating the resulting equation over Ω, and using integration by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega m|u|^r+\int_\Omega r|u|^{r-2}\big(\mu|\nabla u|^2+(\lambda+\mu)|\operatorname{div}u|^2+\mu(r-2)\big|\nabla|u|\big|^2\big)\\&\quad=r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P-r(r-2)(\mu+\lambda)\int_\Omega \operatorname{div}u|u|^{r-3}u\cdot\nabla|u|.\end{aligned}\tag{5.2}$$

For $\varepsilon_1\in(0,1)$, define

$$\phi(\varepsilon_1,r)=\begin{cases}\frac{4\varepsilon_1\mu(r-1)}{3(r-2)(\mu+\lambda)-4\mu}, & \text{if } r>2+\frac{4\mu}{3(\mu+\lambda)},\\ 0, & \text{otherwise.}\end{cases}$$

Case 1:

$$\int_\Omega |u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2\leqslant\phi(\varepsilon_1,r)\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2.\tag{5.3}$$

A direct calculation gives

$$|\nabla u|^2=|u|^2\left|\nabla\left(\frac{u}{|u|}\right)\right|^2+\big|\nabla|u|\big|^2,\tag{5.4}$$

and

$$\operatorname{div}u=|u|\operatorname{div}\left(\frac{u}{|u|}\right)+\frac{u\cdot\nabla|u|}{|u|}.\tag{5.5}$$

By (5.2) and (5.5), we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega m|u|^r+\int_\Omega r|u|^{r-2}\big(\mu|\nabla u|^2+(\lambda+\mu)|\operatorname{div}u|^2+\mu(r-2)\big|\nabla|u|\big|^2\big)\\&\quad=r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P-r(r-2)(\mu+\lambda)\int_\Omega |u|^{r-2}u\cdot\nabla|u|\operatorname{div}\left(\frac{u}{|u|}\right)\\&\qquad-r(r-2)(\mu+\lambda)\int_\Omega |u|^{r-4}\big|u\cdot\nabla|u|\big|^2\\&\quad\leqslant r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P+\frac{r(r-2)(\mu+\lambda)}{4}\int_\Omega |u|^r\left|\operatorname{div}\left(\frac{u}{|u|}\right)\right|^2\\&\quad\leqslant r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P+\frac{3r(r-2)(\mu+\lambda)}{4}\int_\Omega |u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2,\end{aligned}\tag{5.6}$$

where we have used Cauchy inequality. By (5.4) and (5.6), we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega m|u|^r+\mu r\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2+\mu r\int_\Omega |u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2+\mu r(r-2)\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2\\&\quad\leqslant r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P+\frac{3r(r-2)(\mu+\lambda)}{4}\int_\Omega |u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2.\end{aligned}$$

This together with (5.1), (5.3) and Cauchy inequality implies for $\varepsilon>0$

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^r + \mu r(r-1)\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2 \\
&\quad\leqslant r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P + \left(\frac{3r(r-2)(\mu+\lambda)}{4} - \mu r\right)\int_\Omega |u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2 \\
&\quad\leqslant C\int_\Omega m^{\frac{r-2}{2r}}|u|^{r-2}|\nabla u| + \phi(\varepsilon_1, r)\left(\frac{3r(r-2)(\mu+\lambda)}{4} - \mu r\right)\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2 \\
&\quad\leqslant \varepsilon\int_\Omega |u|^{r-2}|\nabla u|^2 + \frac{C}{4\varepsilon}\left(\int_\Omega m|u|^r\right)^{\frac{r-2}{r}} + \phi(\varepsilon_1, r)\left(\frac{3r(r-2)(\mu+\lambda)}{4} - \mu r\right)\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2,
\end{aligned}$$

where we have used $P(m,n) \leqslant Cm^{\frac{1}{2}}$, which can be obtained from (5.1), Lemma 5.1 and the expression of P.

This together with (5.3) and (5.4) gives

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^r + r\left[\mu(r-1) - \phi(\varepsilon_1, r)\left(\frac{3(r-2)(\mu+\lambda)}{4} - \mu\right)\right]\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2 \\
&\quad\leqslant \varepsilon\big[1+\phi(\varepsilon_1, r)\big]\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2 + \frac{C}{4\varepsilon}\left(\int_\Omega m|u|^r\right)^{\frac{r-2}{r}}.
\end{aligned}$$

Taking $\varepsilon = [1+\phi(\varepsilon_1, r)]^{-1} r[\mu(r-1) - \phi(\varepsilon_1, r)(\frac{3(r-2)(\mu+\lambda)}{4} - \mu)]$, we get

$$\frac{d}{dt}\int_\Omega m|u|^r \leqslant \frac{C[1+\phi(\varepsilon_1, r)]}{4\mu r(r-1) - \phi(\varepsilon_1, r)[3r(r-2)(\mu+\lambda) - 4\mu r]}\left(\int_\Omega m|u|^r\right)^{\frac{r-2}{r}}, \tag{5.7}$$

for any $r \in (3, 4]$.

Case 2:

$$\int_\Omega |u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2 > \phi(\varepsilon_1, r)\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2. \tag{5.8}$$

By (5.2), we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^r + \int_\Omega r|u|^{r-2}\big(\mu|\nabla u|^2 + (\lambda+\mu)|\operatorname{div} u|^2 + \mu(r-2)\big|\nabla|u|\big|^2\big) \\
&\quad= r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P - r(r-2)(\mu+\lambda)\int_\Omega \operatorname{div} u|u|^{\frac{r-2}{2}}|u|^{\frac{r-4}{2}}u\cdot\nabla|u| \\
&\quad\leqslant C\int_\Omega m^{\frac{r-2}{2r}}|u|^{r-2}|\nabla u| + r(\mu+\lambda)\int_\Omega |u|^{r-2}|\operatorname{div} u|^2 + \frac{r(r-2)^2(\mu+\lambda)}{4}\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2,
\end{aligned}$$

where we have used Cauchy inequality, (5.1) and $P \leqslant Cm^{\frac{1}{2}}$. Therefore,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^r + \int_\Omega \mu r|u|^{r-2}|\nabla u|^2 + \mu(r-2)r\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2 \\
&\quad\leqslant C\int_\Omega m^{\frac{r-2}{2r}}|u|^{r-2}|\nabla u| + \frac{r(r-2)^2(\mu+\lambda)}{4}\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2.
\end{aligned} \tag{5.9}$$

It follows from (5.4), (5.9) and Cauchy inequality that for any $\varepsilon_0 \in (0, 1)$

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^r+\int_\Omega \mu r|u|^{r-2}\big|\nabla|u|\big|^2+\int_\Omega \mu r|u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2+\mu(r-2)r\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2\\
&\quad\leqslant C\int_\Omega m^{\frac{r-2}{2r}}|u|^{r-2}\big|\nabla|u|\big|+C\int_\Omega m^{\frac{r-2}{2r}}|u|^{r-1}\left|\nabla\left(\frac{u}{|u|}\right)\right|+\frac{r(r-2)^2(\mu+\lambda)}{4}\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2\\
&\quad\leqslant C\int_\Omega m^{\frac{r-2}{2r}}|u|^{r-2}\big|\nabla|u|\big|+\mu r\varepsilon_0\int_\Omega |u|^r\left|\nabla\left(\frac{u}{|u|}\right)\right|^2+\frac{C}{4\mu r\varepsilon_0}\left(\int_\Omega m|u|^r\right)^{\frac{r-2}{r}}\\
&\qquad+\frac{r(r-2)^2(\mu+\lambda)}{4}\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2.
\end{aligned}
$$

Combining (5.8), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^r+r\left[\mu(1-\varepsilon_0)\phi(\varepsilon_1,r)+\mu(r-1)-\frac{(r-2)^2(\mu+\lambda)}{4}\right]\int_\Omega |u|^{r-2}\big|\nabla|u|\big|^2\\
&\quad\leqslant C\int_\Omega m^{\frac{r-2}{2r}}|u|^{r-2}\big|\nabla|u|\big|+\frac{C}{4\mu r\varepsilon_0}\left(\int_\Omega m|u|^r\right)^{\frac{r-2}{r}}.
\end{aligned}\tag{5.10}
$$

(**Subcase 2_1**): If $3\in(2+\frac{4\mu}{3(\mu+\lambda)},\infty)$, i.e. $\mu<3\lambda$, we have for $r\in[3,\infty)$

$$
\phi(\varepsilon_1,r)=\frac{4\varepsilon_1\mu(r-1)}{3(r-2)(\mu+\lambda)-4\mu}.\tag{5.11}
$$

Define

$$
f(\varepsilon_0,\varepsilon_1,r)=\mu(1-\varepsilon_0)\phi(\varepsilon_1,r)+\mu(r-1)-\frac{(r-2)^2(\mu+\lambda)}{4}.\tag{5.12}
$$

By (5.11) and (5.12), we have

$$
f(\varepsilon_0,\varepsilon_1,r)=\frac{4\mu^2(1-\varepsilon_0)\varepsilon_1(r-1)}{3(r-2)(\mu+\lambda)-4\mu}+\mu(r-1)-\frac{(r-2)^2(\mu+\lambda)}{4},\tag{5.13}
$$

for $r\in[3,\infty)$. Particularly,

$$
f(0,1,3)=\frac{8\mu^2}{3\lambda-\mu}+\frac{7\mu-\lambda}{4}>0.
$$

Here we have used $\frac{\mu}{3}<\lambda<\frac{25}{3}\mu$.

Since $f(\varepsilon_0,\varepsilon_1,r)$ is continuous w.r.t. $(\varepsilon_0,\varepsilon_1,r)$ in $(0,1)\times(0,1)\times[3,\infty)$, there exists $\varepsilon_0,\varepsilon_1\in(0,1)$ and $r_1\in(3,4]$ such that

$$
f(\varepsilon_0,\varepsilon_1,r_1)>0.
$$

From (5.10), Cauchy inequality and Hölder inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^{r_1}+r_1 f(\varepsilon_0,\varepsilon_1,r_1)\int_\Omega |u|^{r_1-2}\big|\nabla|u|\big|^2\\
&\quad\leqslant r_1 f(\varepsilon_0,\varepsilon_1,r_1)\int_\Omega |u|^{r_1-2}\big|\nabla|u|\big|^2+\frac{C}{4r_1 f(\varepsilon_0,\varepsilon_1,r_1)}\left(\int_\Omega m|u|^{r_1}\right)^{\frac{r_1-2}{r_1}}+\frac{C}{4\mu r_1\varepsilon_0}\left(\int_\Omega m|u|^{r_1}\right)^{\frac{r_1-2}{r_1}}.
\end{aligned}
$$

Therefore,

$$\frac{d}{dt}\int\limits_\Omega m|u|^{r_1} \leqslant C\left[\frac{1}{f(\varepsilon_0,\varepsilon_1,r_1)}+\frac{1}{\mu\varepsilon_0}\right]\left(\int\limits_\Omega m|u|^{r_1}\right)^{\frac{r_1-2}{r_1}}. \tag{5.14}$$

(**Subcase 2₂**): $3 \notin (2+\frac{4\mu}{3(\mu+\lambda)},\infty)$, i.e. $\mu \geqslant 3\lambda$. In this case, it is easy to verify that the following inequality holds for any $r \in (3,4]$

$$r\left[\mu(1-\varepsilon_0)\phi(\varepsilon_1,r)+\mu(r-1)-\frac{(r-2)^2(\mu+\lambda)}{4}\right] > 2\mu. \tag{5.15}$$

We obtain from (5.10), (5.15), Cauchy inequality and Hölder inequality

$$\begin{aligned}\frac{d}{dt}\int\limits_\Omega m|u|^r + 2\mu\int\limits_\Omega |u|^{r-2}\big|\nabla|u|\big|^2 &\leqslant C\int\limits_\Omega m^{\frac{r-2}{2r}}|u|^{r-2}\big|\nabla|u|\big| + \frac{C}{4\mu r\varepsilon_0}\left(\int\limits_\Omega m|u|^r\right)^{\frac{r-2}{r}}\\ &\leqslant 2\mu\int\limits_\Omega |u|^{r-2}\big|\nabla|u|\big|^2+\frac{C}{\mu\varepsilon_0}\left(\int\limits_\Omega m|u|^r\right)^{\frac{r-2}{r}}.\end{aligned}$$

This implies

$$\frac{d}{dt}\int\limits_\Omega m|u|^r \leqslant \frac{C}{\mu\varepsilon_0}\left(\int\limits_\Omega m|u|^r\right)^{\frac{r-2}{r}}. \tag{5.16}$$

Particularly, (5.16) is also valid for $r=r_1$.

Summarily, for Case 2, we obtain from (5.14) and (5.16)

$$\frac{d}{dt}\int\limits_\Omega m|u|^{r_1} \leqslant C\left(\int\limits_\Omega m|u|^{r_1}\right)^{\frac{r_1-2}{r_1}}, \tag{5.17}$$

for some constant C, if $\lambda < \frac{25}{3}\mu$ and (5.8) is valid.

It follows from (5.7) and (5.17) that

$$\frac{d}{dt}\int\limits_\Omega m|u|^{r_1} \leqslant C\left(\int\limits_\Omega m|u|^{r_1}\right)^{\frac{r_1-2}{r_1}}, \tag{5.18}$$

for some constant C, if $\lambda < \frac{25}{3}\mu$.

Since $\frac{r_1-2}{r_1} \in (0,1)$, we complete the proof of Lemma 5.2 after using Young inequality and Gronwall inequality in (5.18) and still denoting r_1 by r. □

Similar to Lemma 4.2, and Corollary 4.5, we get the next lemma.

Lemma 5.3. *Under the conditions of Theorem* 1.2 *and* (5.1)*, for* $0 \leqslant T < T^*$*, we have*

$$\begin{cases}\|P\|_{L^\infty(Q_T)}+\|P_m\|_{L^\infty(Q_T)}+\|P_n\|_{L^\infty(Q_T)} \leqslant C,\\ \|\nabla u\|_{L^2(0,T;L^6)}+\|\nabla u\|_{L^\infty(0,T;L^2)} \leqslant C.\end{cases} \tag{5.19}$$

Here we have used (5.1) and Lemma 5.2.

Lemma 5.4. *Under the conditions of Theorem* 1.2 *and* (5.1)*, for* $0 \leqslant T < T^*$*, we have*

$$\int\limits_\Omega m|\dot u|^2 + \int\limits_0^T\int\limits_\Omega |\nabla \dot u|^2 \leqslant C,$$

where $\dot u = u_t + u\cdot\nabla u$.

Proof. Similar to Lemma 4.6, we have

$$\begin{aligned}\int_\Omega m|\dot u|^2+\int_0^t\int_\Omega\big(\mu|\nabla\dot u|^2+(\mu+\lambda)|\operatorname{div}\dot u|^2\big)&\leqslant C+C\int_0^t\big[\big(1+\|\sqrt{m}\dot u\|_{L^2}\big)\|\nabla u\|_{L^6}^2\big]\\&\leqslant C+C\int_0^t\big(\|\sqrt{m}\dot u\|_{L^2}^2\|\nabla u\|_{L^6}^2\big),\end{aligned}$$

where we have used (5.19) and Cauchy inequality. This together with Gronwall inequality and (5.19) completes the proof of Lemma 5.4. □

Denote $w=u-h$, where h is the solution to

$$\begin{cases}Lh=\nabla P(m,n), & \text{in } \Omega\times(0,T],\\ h|_{\partial\Omega}=0.\end{cases}\tag{5.20}$$

Similar to (4.26), we have

$$\begin{cases}Lw=m\dot u, & \text{in } \Omega\times(0,T],\\ w|_{\partial\Omega}=0.\end{cases}\tag{5.21}$$

Due to (5.1), (5.21), Lemma 2.3 and Lemma 5.4, we immediately give the following result.

Corollary 5.5. *Under the conditions of Theorem* 1.2 *and* (5.1)*, for* $0\leqslant T<T^*$*, we have*

$$\|w\|_{L^2(0,T;W^{2,q})}\leqslant C.$$

In the following, we give the estimates of the derivatives of m and n.

Lemma 5.6. *Under the conditions of Theorem* 1.2 *and* (5.1)*, for* $0\leqslant T<T^*$*, we have*

$$\sup_{t\in[0,T]}\big\|(\nabla m,\nabla n)(t)\big\|_{L^q}\leqslant C.$$

Proof. Differentiating Eq. $(1.1)_1$ with respect to x_i, then multiplying both sides of the resulting equation by $q|\partial_i m|^{q-2}\partial_i m$, we get

$$\begin{aligned}&\partial_t|\partial_i m|^q+\operatorname{div}\big(|\partial_i m|^q u\big)+(q-1)|\partial_i m|^q\operatorname{div}u\\&\quad+qm|\partial_i m|^{q-2}\partial_i m\partial_i\operatorname{div}u+q|\partial_i m|^{q-2}\partial_i m\partial_i u\cdot\nabla m=0.\end{aligned}\tag{5.22}$$

Integrating (5.22) over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega|\nabla m|^q&\leqslant C\int_\Omega|\nabla u||\nabla m|^q+q\int_\Omega m|\nabla\operatorname{div}u||\nabla m|^{q-1}\\&\leqslant C\|\nabla u\|_{L^\infty}\|\nabla m\|_{L^q}^q+C\big\|\nabla^2u\big\|_{L^q}\|\nabla m\|_{L^q}^{q-1}.\end{aligned}\tag{5.23}$$

Similarly,

$$\frac{d}{dt}\int_\Omega|\nabla n|^q\leqslant C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^q}^q+C\big\|\nabla^2u\big\|_{L^q}\|\nabla n\|_{L^q}^{q-1}.\tag{5.24}$$

By (5.23)–(5.24), we have

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla m\|_{L^q}+\|\nabla n\|_{L^q}\big)\\&\quad\leqslant C\big(1+\|\nabla w\|_{L^\infty}+\|\nabla h\|_{L^\infty}\big)\big(\|\nabla m\|_{L^q}+\|\nabla n\|_{L^q}\big)+C\big\|\nabla^2w\big\|_{L^q}+C\big\|\nabla^2h\big\|_{L^q}.\end{aligned}\tag{5.25}$$

Since P_m and P_n are bounded, we have from (5.20) and Lemma 2.3

$$\left\|\nabla^2 h\right\|_{L^q} \leqslant C\big(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}\big). \tag{5.26}$$

Applying (5.25)–(5.26) and Sobolev inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\big(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}\big)\\
&\quad \leqslant C\big(1 + \|w\|_{W^{2,q}} + \|\nabla h\|_{L^\infty}\big)\big(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}\big) + C\left\|\nabla^2 w\right\|_{L^q}.
\end{aligned} \tag{5.27}$$

Using (5.19), (5.20) and Lemmas 2.3–2.5, we have

$$\begin{aligned}
\|\nabla h\|_{L^\infty} &\leqslant C\big(1 + \|\nabla h\|_{BMO(\Omega)} \ln\big(e + \left\|\nabla^2 h\right\|_{L^q}\big)\big)\\
&\leqslant C\big(1 + \|P\|_{L^\infty \cap L^2} \ln\big(e + \|\nabla P\|_{L^q}\big)\big)\\
&\leqslant C\big(1 + \ln\big(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}\big)\big).
\end{aligned} \tag{5.28}$$

From (5.27)–(5.28) and Cauchy inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\big(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}\big)\\
&\quad \leqslant C\big(1 + \|w\|_{W^{2,q}} + \ln\big(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}\big)\big)\big(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}\big) + C\left\|\nabla^2 w\right\|_{L^q}.
\end{aligned} \tag{5.29}$$

Denote $G(t) = e + \|\nabla m(t)\|_{L^q} + \|\nabla n(t)\|_{L^q}$, we have from (5.29)

$$\frac{d}{dt}G(t) \leqslant C\left\|\nabla^2 w\right\|_{L^q} + C\big(1 + \|w\|_{W^{2,q}}\big)G(t) + CG(t)\ln G(t). \tag{5.30}$$

Multiplying (5.30) by $\frac{1}{G(t)}$, and using $G > 1$, we have

$$\frac{d}{dt}\ln G(t) \leqslant C\big(1 + \|w\|_{W^{2,q}}\big) + C\ln G(t). \tag{5.31}$$

Using Gronwall inequality, Corollary 5.5 and (5.31), we complete the proof of Lemma 5.6. □

Lemma 5.7. *Under the conditions of Theorem* 1.2 *and* (5.1)*, for* $0 \leqslant T < T^*$*, we have*

$$\|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;W^{2,q})} \leqslant C.$$

Proof. Rewrite $(1.1)_3$ as

$$Lu = m\dot{u} + \nabla P(m, n).$$

By (5.1), (5.19), Lemmas 2.3, 5.4 and 5.6, we have

$$\begin{aligned}
\|u\|_{H^2} &\leqslant C\big(\|m\dot{u}\|_{L^2} + \|\nabla P\|_{L^2}\big)\\
&\leqslant C\big(\left\|m^{\frac{1}{2}}\dot{u}\right\|_{L^2} + \|\nabla m\|_{L^2} + \|\nabla n\|_{L^2}\big) \leqslant C,
\end{aligned}$$

and

$$\begin{aligned}
\|u\|_{L^2(0,T;W^{2,q})} &\leqslant C\big(\|m\dot{u}\|_{L^2(0,T;L^q)} + \|\nabla P\|_{L^2(0,T;L^q)}\big)\\
&\leqslant C\big(\|\nabla\dot{u}\|_{L^2(0,T;L^2)} + \|\nabla m\|_{L^2(0,T;L^q)} + \|\nabla n\|_{L^2(0,T;L^q)}\big) \leqslant C. \qquad \square
\end{aligned}$$

By (5.1), Lemma 5.4, Lemma 5.7 and Sobolev inequality, we get the following result.

Corollary 5.8. *Under the conditions of Theorem* 1.2 *and* (5.1)*, for* $0 \leqslant T < T^*$*, we have*

$$\int_\Omega m|u_t|^2 + \int_0^T\int_\Omega |\nabla u_t|^2 \leqslant C.$$

By (5.1), Lemmas 5.3, 5.6, 5.7 and Corollary 5.8, we know that T^* is not the maximal existence time for the strong solution $(m, n, u)(x, t)$ to the problem (1.1)–(1.5). This is a contradiction with the definition of T^*. Therefore, (5.1) is invalid, i.e.,

$$\lim_{T \to T^*} \sup \left\| m(t) \right\|_{L^\infty(0,T;L^\infty)} = \infty.$$

The proof of Theorem 1.2 is completed. $\square$

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